Typed Kleene Algebra with Products and Iteration Theories

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***Abstract*—We develop a typed equational system that subsumes both iteration theories and typed Kleene algebra in a common framework. Our approach is based on cartesian categories endowed with commutative strong monads to handle nondeterminism.**

I. Introduction

In the realm of equational systems for reasoning about iteration, two chief complementary bodies of work stand out. One of these is *iteration theories* (IT), the subject of the extensive monograph of Bloom and Ésik [1] as well as many other authors (see the cited literature). The primary motivation for iteration theories is to capture in abstract form the equational properties of iteration on structures that arise in domain theory and program semantics, such as continuous functions on ordered sets. Of central interest is the dagger operation †, a kind of parameterized least fixpoint operator, that when applied to an object representing a simultaneous system of equations gives an object representing the least solution of those equations. Much of the work on iteration theories involves axiomatizing or otherwise characterizing the equational theory of iteration as captured by †. Complete axiomatizations have been provided [2–4] as well as other algebraic and categorical characterizations [5, 6].

Bloom and Ésik claim that "…the notion of an iteration theory seems to axiomatize the equational properties of all computationally interesting structures…" [7]. This is true to a certain extent, certainly if one is interested only in structures that arise in domain theory and programming language semantics. However, it is not the entire story.

Another approach to equational reasoning about iteration that has met with some success over the years is the notion of *Kleene algebra* (KA), the algebra of regular expressions. KA has a long history going back to the original paper of Kleene [8] and was further developed by Conway, who coined the name Kleene algebra in his 1971 monograph [9]. It has since been studied by many authors. KA relies on an iteration operator * that characterizes iteration in a different way from †. Its principal models are not those of domain theory, but rather basic algebraic objects such as sets of strings (in which * gives the Kleene asterate operation), binary relations (in which * gives reflexive transitive closure), and other structures with applications in shortest path algorithms on graphs and geometry of convex sets. Complete axiomatizations and complexity analyses have been given; the regular sets of strings over an alphabet A form the free KA on generators A in much the same way that the rational $\Sigma_\perp$-trees form the free IT on a signature Σ.

Although the two systems fulfill many of the same objectives and are related at some level, there are many technical and stylistic differences. Whereas iteration theories are based on Lawvere theories, a categorical concept, Kleene algebra operates primarily at a level of abstraction one click down. For this reason, KA may be somewhat more accessible. KA has been shown to be useful in several nontrivial static analysis and verification tasks (see e.g. [10, 11]). Also, KA can model nondeterministic computation, whereas IT is primarily deterministic.

Nevertheless, both systems have claimed to capture the notion of iteration in a fundamental way, and it is interesting to ask whether they can somehow be reconciled. This is the investigation that we have undertaken in this paper. We start with the observation that ITs use the objects of a category to represent types. Technically the objects of interest in ITs are morphisms $f : n \to m$ in a category whose objects are natural numbers, and the morphism $f : n \to m$ is meant to model functions $f : A^m \to A^n$ (the arrows are reversed for technical reasons). Thus ITs might be captured by a version of KA with types. Although the primary version of KA is untyped, there is a notion of typed KA [12], although it only has types of the form $A \to B$, whereas to subsume IT it would need products as well. The presence of products allow ITs to capture parameterized fixpoints through the rule

$$\frac{f : n \to n + m}{f^\dagger : n \to m}$$

giving the parameterized least fixpoint $f^\dagger : A^m \to A^n$ of a parameterized function $f : A^m \times A^n \to A^n$. This would be possible to capture in KA if the typed version had products, which it does not. On the other hand, KA allows the modeling of nondeterministic computation, which IT does not, at least not in any obvious way. Thus to capture both systems, it would seem that we need to extend the type system of typed KA, or extend the categorical framework of IT to handle nondeterminism, or both.

The result of our investigation is a common categorical framework based on cartesian categories (categories with products) combined with a monadic treatment of nondeterminism. Types are represented by objects in the category, and we identify the appropriate axioms in the form of typed equations that allow equational reasoning on the morphisms. Our framework captures iteration as represented in ITs and KAs in a common language. We show how to define the KA operations as enrichments on the morphisms and how to define † in terms of *.

Our main contributions are as follows.

- **Commutative strong monads.** To accommodate nondeterminism, we need to lift the computation to the Kleisli category of a monad representing nondeterminate values. However, the ordinary powerset monad does not suffice for this purpose, as it does not interact well with non-strictness. We axiomatize the relevant properties for an arbitrary *commutative strong monad*, where (i) the property of *commutativity* captures the idea that the computation of a pair can be done in either order, and (ii) *strength* refers to tensorial strength, which axiomatizes the interaction of pairs with the nondeterminism monad.
- **Lazy pairs**. We need to model non-strict (lazy) evaluation of programs in the presence of products. Ordinarily, a pair $\langle x, \bot \rangle$ would be $\bot$ by strictness. This requires the development of the concept of *lazy pairs* and its categorical axiomatization. Intuitively, in the case of *eager pairs*, the computation of a pair $\langle v, \bot \rangle$, where v is a value and $\bot$ denotes a diverging computation, would also be diverging, i.e., $\langle v, \bot \rangle = \bot$. This makes it impossible to recover the left component v of the pair: $\langle v, \bot \rangle ; \pi_1 = \bot$.
- **Simplified axiomatization of commutative strong monads and lazy pairs**. We have given a simplified axiomatization of commutative monads with lazy pairs in terms of a certain operator ψ that captures the interaction of these concepts in a very concise form, much simpler than the axiomatizations of the two of them separately. This is an adaptation of a construction that can be found in the work of Kock in the 1970s [13, 14]. We use this extensively in our development to simplify arguments.
- **Deterministic arrows**. Certain properties work only for deterministic computations. We show how to capture the necessary properties of determinism in the Kleisli category. A separate syntactic arrow provides a convenient notation for reasoning about deterministic computation in the underlying category when working in the Kleisli category, and we provide an axiomatization of the necessary properties.
- **Lifting the cartesian structure**. We show how the cartesian structure (pairing and projections) in the underlying category can be lifted in a smooth way to corresponding operations in the Kleisli category.
- **Capturing nondeterminism**. We give three equivalent ways of capturing (angelic) nondeterminism in the homsets of the Kleisli category of a monad. These characterizations make essential use of cartesian structure of the base category.
- **Capturing IT and KA**. We show how to enrich the homsets of the Kleisli category with the KA operations, including *, to obtain typed KA with products, and that all the axioms of KA (except the strictness axiom) are satisfied. Sequential composition is modeled by Kleisli composition. We also show that Park theories [4] are subsumed by KA with products. This is our main result.
- **Model theory**. Finally, we show that two particular monads, the *lowerset monad* and the *ideal completion monad*, provide natural concrete models in that they are commutative strong monads with lazy pairs. The ideal completion monad involves ideal completion in ω-complete partial orders (ω-CPOs) and models nondeterminism in those structures.

A detailed account of related work is given in §IX.

II. Cartesian Category with Bottom Elements

In order to model non-strict (lazy) evaluation of programs and lazy pairs we consider our base category to be a *cartesian category* $\mathcal{C}$ *with bottom elements*. For an object X, we write the identity for X as $\mathrm{id}_X : X \to X$. The composition operation is written as ; and the operands are given in diagrammatic order.

$$\frac{f : X \to Y \qquad g : Y \to Z}{f ; g : X \to Z}.$$

For objects X and Y, we denote by $X \times Y$ their product with corresponding left and right projections $\pi_1^{XY} : X \times Y \to X$ and $\pi_2^{XY} : X \times Y \to Y$ respectively. The typing rule for the pairing operation $\langle \cdot, \cdot \rangle$ is

$$\frac{f : X \to Y \qquad g : X \to Z}{\langle f, g \rangle : X \to Y \times Z}.$$

The terminal object is denoted by $\mathbb{1}$. We write $\bot_X : X \to \mathbb{1}$ for the unique arrow from X to $\mathbb{1}$. The typed equational axioms

$$\frac{f : X \to Y \qquad g : X \to Z}{\langle f, g \rangle ; \pi_1 = f : X \to Y} \qquad \frac{f : X \to Y \qquad g : X \to Z}{\langle f, g \rangle ; \pi_2 = g : X \to Y}$$

$$\frac{h : X \to Y \times Z}{\langle h ; \pi_1, h ; \pi_2 \rangle = h : X \to Y \times Z} \qquad \frac{f : X \to \mathbb{1}}{f = \bot_X : X \to \mathbb{1}}$$

say that the operations $\times, \pi_1, \pi_2, \langle \cdot, \cdot \rangle, \mathbb{1}, \bot$ endow $\mathcal{C}$ with cartesian structure. For every object X, the *bottom element* of X is written as $\bot_{\mathbb{1}X} : \mathbb{1} \to X$. Define the *bottom morphism* $\bot_{XY}$ from X to Y by $\bot_{XY} := \bot_X ; \bot_{\mathbb{1}Y}$. The bottom morphisms satisfy the axiom

$$\frac{f : X \to Y}{f ; \bot_{YZ} = \bot_{XZ} : X \to Z}.$$

A morphism $f : X \to Y$ satisfiying $\bot_{\mathbb{1}X} ; f = \bot_{\mathbb{1}Y}$ is called *strict*.

Remark 1: Let $\mathcal{C}$ be a category with binary products given by $\pi_1, \pi_2, \langle \cdot, \cdot \rangle$. We define the *product functor* $\times$ from $\mathcal{C} \times \mathcal{C}$ to $\mathcal{C}$ by $(X, Y) \mapsto X \times Y$ and

$$\frac{f_1 : X_1 \to Y_1 \qquad f_2 : X_2 \to Y_2}{f_1 \times f_2 := \langle \pi_1 ; f_1, \pi_2 ; f_2 \rangle : X_1 \times X_2 \to Y_1 \times Y_2}.$$

$$\begin{tikzcd}
PX \times PY \arrow[r, "t_{PX,Y}"] \arrow[d, "\tau_{X,PY}"'] \arrow[rrd, "\psi_{X,Y}"] & P(PX \times Y) \arrow[r, "P\tau_{X,Y}"] & P^2(X \times Y) \arrow[d, "\mu_{X\times Y}"] \\
P(X \times PY) \arrow[r, "Pt_{X,Y}"'] & P^2(X \times Y) \arrow[r, "\mu_{X\times Y}"'] & P(X \times Y)
\end{tikzcd}$$

Fig. 1. Commutativity axiom for the strong monad $(P, \eta, \mu), t$.

The fact that $\times$ is a functor, as well as the equations $f; \langle g, h\rangle = \langle f; g, f; h\rangle$ and $\langle f_1; g_1, f_2; g_2\rangle = \langle f_1, f_2\rangle; (g_1 \times g_2)$, can be easily proved with equational reasoning.

III. Commutative Strong Monads with Lazy Pairs

For this section, we consider a cartesian category $\mathcal{C}$, whose cartesian structure is explicitly given by $\times, \pi_1, \pi_2, \langle\cdot,\cdot\rangle, \mathbb{1}, \bot$.

A *monad* (P, η, μ) over $\mathcal{C}$ consists of the following: An endofunctor $P : \mathcal{C} \to \mathcal{C}$, and natural transformations $\eta_X : X \to PX$ and $\mu_X : P^2X \to PX$, called the *unit* and *multiplication* of the monad respectively. Additionally, they satisfy the equations: $P\mu_X; \mu_X = \mu_{PX}; \mu_X$, $\eta_{PX}; \mu_X = \mathrm{id}_{PX}$, and $P\eta_X; \mu_X = \mathrm{id}_{PX}$.

A monad (P, η, μ) is *strong* with *tensorial strength* $t_{X,Y} : X \times PY \to P(X \times Y)$ if t is a natural transformation satisfying the axioms given by the commutative diagrams of Figure 8 (in the Appendix). The arrow $\alpha_{X,Y,Z} : (X \times Y) \times Z \to X \times (Y \times Z)$ is the natural isomorphism defined by $\alpha := \langle\pi_1; \pi_1, \langle\pi_1; \pi_2, \pi_2\rangle\rangle$. We define the *dual tensorial strength* $\tau_{X,Y} : PX \times Y \to P(X \times Y)$ as the composite of

$$PX \times Y \xrightarrow{\mathsf{s}_{PX,Y}} Y \times PX \xrightarrow{t_{Y,X}} P(Y \times X) \xrightarrow{P\mathsf{s}_{Y,X}} P(X \times Y),$$

where $\mathsf{s}_{X,Y} : X \times Y \to Y \times X$ is the natural isomorphism given by $\mathsf{s}_{X,Y} = \langle\pi_2, \pi_1\rangle$. The properties satisfied by t imply that τ is also a natural transformation and the diagrams of Figure 9 (in the Appendix) commute. The arrow $\beta_{X,Y,Z} : X \times (Y \times Z) \to (X \times Y) \times Z$ is the natural isomorphism which is the inverse of $\alpha_{X,Y,Z}$. For all objects X, Y define the morphism $\psi_{X,Y} : PX \times PY \to P(X \times Y)$ as the composite of

$$\begin{tikzcd}
PX \times PY \arrow[r, bend left, "t_{PX,Y}"] & P(PX \times Y) \arrow[r, bend left, "P\tau_{X,Y}"] & P^2(X \times Y) \arrow[r, bend left, "\mu_{X\times Y}"] & P(X \times Y).
\end{tikzcd}$$

Using the fact that t and τ are natural transformations, it can be shown that ψ is also a natural transformation.

A strong monad $(P, \eta, \mu), t$ is *commutative* if the diagram of Figure 1 commutes. Intuitively, the commutativity condition says that when computing a pair it does not matter in which order the components are computed.

We are interested in strong monads that model *lazy pairs*. In the case of *eager pairs*, the computation of a pair $\langle v, \bot\rangle$, where v is a value and $\bot$ denotes a diverging computation, would also be diverging, i.e., $\langle v, \bot\rangle = \bot$. Therefore, it is not possible to recover the left component v of the pair: $\langle v, \bot\rangle; \pi_1 = \bot$. So, for lazy pairs we need to stipulate an additional axiom,

$$\begin{tikzcd}
X \times PY \arrow[r, "t"] \arrow[d, "\pi_1"'] & P(X \times Y) \arrow[d, "P\pi_1"] \\
X \arrow[r, "\eta"] & PX
\end{tikzcd}
\qquad
\begin{tikzcd}
PX \times Y \arrow[r, "\tau"] \arrow[d, "\pi_2"'] & P(X \times Y) \arrow[d, "P\pi_2"] \\
Y \arrow[r, "\eta"] & PY
\end{tikzcd}$$

Fig. 2. Lazy pairs axiom in terms of t and equivalently in terms of τ.

$$\begin{tikzcd}
PX \times PY \arrow[r, "\psi"] \arrow[dr, "\mathrm{id}"'] & P(X \times Y) \arrow[d, "\langle P\pi_1, P\pi_2\rangle"] \\
& PX \times PY
\end{tikzcd}
\qquad
\begin{tikzcd}
X \times Y \arrow[r, "\eta \times \eta"] \arrow[dr, "\eta"'] & PX \times PY \arrow[d, "\psi"] \\
& P(X \times Y)
\end{tikzcd}$$

$$\begin{tikzcd}
(PX \times PY) \times PZ \arrow[r, "\psi \times \mathrm{id}"] \arrow[d, "\alpha"'] & P(X \times Y) \times PZ \arrow[r, "\psi"] & P((X \times Y) \times Z) \arrow[d, "P\alpha"] \\
PX \times (PY \times PZ) \arrow[r, "\mathrm{id} \times \psi"'] & PX \times P(Y \times Z) \arrow[r, "\psi"'] & P(X \times (Y \times Z))
\end{tikzcd}$$

$$\begin{tikzcd}
P^2X \times P^2Y \arrow[r, "\psi_{PX,PY}"] \arrow[d, "\mu_X \times \mu_Y"'] & P(PX \times PY) \arrow[r, "P\psi_{X,Y}"] & P^2(X \times Y) \arrow[d, "\mu_{X\times Y}"] \\
PX \times PY \arrow[rr, "\psi_{X,Y}"] & & P(X \times Y)
\end{tikzcd}$$

$$\begin{tikzcd}
PX \times PY \arrow[r, "\mathsf{s}"'] \arrow[rrr, bend left, "\psi"] & PY \times PX \arrow[r, "\psi"'] & P(Y \times X) \arrow[r, "P\mathsf{s}"'] & P(X \times Y)
\end{tikzcd}$$

Fig. 3. Commutative strong monad with lazy pairs (given in terms of ψ).

which can be given equivalently in terms of t or τ. The "lazy pairs" axiom says that the diagrams of Figure 2 commute.

Definition 2: Let $\mathcal{C}$ be a category with explicitly given cartesian structure. We say that $(P, \eta, \mu), t$ is a *commutative strong monad with lazy pairs* if (P, η, μ) is a monad with tensorial strength t so that the commutativity axiom of Figure 1 and the lazy pairs axiom of Figure 2 hold.

Remark 3: Now, we will discuss how a commutative monad with lazy pairs can be equivalently given in terms of ψ. In Figure 3 we give properties that ψ satisfies, when it is defined in terms of t as before [13]. Conversely, we consider a monad (P, η, μ) over $\mathcal{C}$ together with a natural transformation $\psi : PX \times PY \to P(X \times Y)$ satisfying the axioms of Figure 3. Then, we can define t as the composite of

$$X \times PY \xrightarrow{\eta_X \times \mathrm{id}_{PY}} PX \times PY \xrightarrow{\psi_{X,Y}} P(X \times Y)$$

and recover all the axioms we had given for $(P, \eta, \mu), t$ [14].

A. Kleisli Category, Unit Functor, Deterministic Arrows

Let (P, η, μ) be a monad over $\mathcal{C}$. The *Kleisli category* $\mathcal{C}_P$ has the same objects as $\mathcal{C}$. For all objects X, Y the homset $\mathcal{C}_P(X, Y)$ is equal to the homset $\mathcal{C}(X, PY)$. We use the notation $f : X \rightharpoonup Y$ for an arrow in $\mathcal{C}_P(X, Y)$, which is also an arrow $f : X \to PY$ in $\mathcal{C}(X, PY)$. The composition operation in $\mathcal{C}_P$ is the *Kleisli composition* operation, denoted $\boldsymbol{;}$, which is defined as

$$\frac{f : X \rightharpoonup Y \qquad g : Y \rightharpoonup Z}{f \boldsymbol{;} g := f; Pg; \mu_Z : X \rightharpoonup Z}.$$

For object X, the identity in $\mathcal{C}_P$ is $\eta_X : X \rightharpoonup X$. The equations

$$\frac{f : X \rightharpoonup Y}{\eta_X ; f = f : X \rightharpoonup Y} \qquad \frac{f : X \rightharpoonup Y}{f ; \eta_Y = f : X \rightharpoonup Y} \tag{1}$$

$$\frac{f : X \rightharpoonup Y \qquad g : Y \rightharpoonup Z \qquad h : Z \rightharpoonup W}{(f ; g) ; h = f ; (g ; h) : X \rightharpoonup W} \tag{2}$$

stating that $\mathcal{C}_P$ is a category can be shown from the definitions and the monad axioms.

Remark 4: For $f : X \to Y$ and $g : Y \to PZ$, it holds that $f; g = (f; \eta_Y) ; g$. Indeed, we have that $(f; \eta_Y) ; g = f; \eta_Y; Pg; \mu_Z = f; g; \eta_{PZ}; \mu_Z = f; g$, using the fact that η is natural and one of the monad axioms.

Definition 5 (unit functor): We define the map $H = (-; \eta)$ from the category $\mathcal{C}$ to the Kleisli category $\mathcal{C}_P$ as follows:

$$H : X \mapsto X \qquad \frac{f : X \to Y}{Hf := f; \eta_Y : X \rightharpoonup Y}$$

We verify that H is a functor. First, note that it sends the identity $\mathrm{id}_X : X \to X$ of $\mathcal{C}$ to the identity $H\mathrm{id}_X = \eta_X : X \rightharpoonup X$ of $\mathcal{C}_P$. Moreover, the rule

$$\frac{f : X \to Y \qquad g : Y \to Z}{H(f; g) = Hf ; Hg : X \rightharpoonup Z}$$

is valid.

$$\begin{aligned}
Hf ; Hg &= f; \eta_Y; P(g; \eta_Z); \mu_Z && \text{[def.]}\\
&= f; \eta_Y; Pg; P\eta_Z; \mu_Z && [P \text{ functor}]\\
&= f; \eta_Y; Pg && [P \text{ monad}]\\
&= f; g; \eta_Z && [\eta \text{ natural}]\\
&= H(f; g). && [H \text{ def}]
\end{aligned}$$

So, H is a functor from $\mathcal{C}$ to $\mathcal{C}_P$. We call this the *unit functor* of the monad.

Definition 6 (deterministic arrows): We say that an arrow $f : X \rightharpoonup Y$ of $\mathcal{C}_P$ is a *deterministic arrow* if there exists an arrow $f' : X \to Y$ of $\mathcal{C}$ such that $f = f'; \eta_Y = Hf' : X \to PY$. So, the deterministic arrows of the Kleisli category are exactly the image of the arrows of $\mathcal{C}$ under the unit functor H. We indicate that f is a deterministic arrow of $\mathcal{C}_P$ by writing $f : X \twoheadrightarrow Y$. The Kleisli composite of two deterministic arrows is also a deterministic arrow:

$$\frac{f : X \twoheadrightarrow Y \qquad g : Y \twoheadrightarrow Z}{f ; g : X \twoheadrightarrow Z}. \tag{3}$$

Suppose that $f = Hf' : X \twoheadrightarrow Y$ and $g = Hg' : Y \twoheadrightarrow Z$ are deterministic arrows. Then, $f ; g = Hf' ; Hg' = H(f'; g')$ since H is a functor. So, $f ; g$ is a deterministic arrow. The identity $\eta_X : X \twoheadrightarrow X$ is a deterministic arrow because $\eta_X = H\mathrm{id}_X$.

B. Kleisli Pairing, Projections, Product

Let $(P, \eta, \mu), t$ be a commutative strong monad over $\mathcal{C}$ with lazy pairs. In this section we will define "Kleisli versions" of projections, the pairing operation, and the product functor. We will prove useful properties that they satisfy. The notion of deterministic arrow turns out to be relevant.

Definition 7 (Kleisli pairing and projections): We define the *Kleisli pairing* operation $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ in $\mathcal{C}_P$ by

$$\frac{f : X \rightharpoonup Y \qquad g : X \rightharpoonup Z}{\langle\!\langle f, g \rangle\!\rangle := \langle f, g \rangle; \psi : X \rightharpoonup Y \times Z}.$$

We also define the *Kleisli projections* (left and right):

$$\begin{aligned}
\varpi_1 &:= \pi_1; \eta = H\pi_1 : X \times Y \twoheadrightarrow X\\
\varpi_2 &:= \pi_2; \eta = H\pi_2 : X \times Y \twoheadrightarrow Y
\end{aligned}$$

We note that the Kleisli projections are deterministic arrows.

We claim that if $f : X \twoheadrightarrow Y$ and $g : X \twoheadrightarrow Z$ are deterministic arrows, then so is $\langle\!\langle f, g \rangle\!\rangle : X \twoheadrightarrow Y \times Z$:

$$\frac{f : X \twoheadrightarrow Y \qquad g : X \twoheadrightarrow Z}{\langle\!\langle f, g \rangle\!\rangle : X \twoheadrightarrow Y \times Z}. \tag{4}$$

This is an immediate consequence of the following rule, which states that H commutes with the pairing operation:

$$\frac{f : X \to Y \qquad g : X \to Z}{H\langle f, g \rangle = \langle\!\langle Hf, Hg \rangle\!\rangle : X \rightharpoonup Y \times Z}.$$

This can be seen by: $\langle\!\langle Hf, Hg \rangle\!\rangle = \langle f; \eta, g; \eta \rangle; \psi = \langle f, g \rangle; (\eta \times \eta); \psi = \langle f, g \rangle; \eta = H\langle f, g \rangle$.

Theorem 8: The following typed equations for Kleisli projections/pairing are valid:

$$\frac{f : X \rightharpoonup Y \qquad g : X \rightharpoonup Z}{\langle\!\langle f, g \rangle\!\rangle ; \varpi_1 = f : X \rightharpoonup Y} \tag{5}$$

$$\frac{f : X \rightharpoonup Y \qquad g : X \rightharpoonup Z}{\langle\!\langle f, g \rangle\!\rangle ; \varpi_2 = g : X \rightharpoonup Z} \tag{6}$$

$$\frac{h : X \twoheadrightarrow Y \times Z}{\langle\!\langle h ; \varpi_1, h ; \varpi_2 \rangle\!\rangle = h : X \twoheadrightarrow Y \times Z} \tag{7}$$

$$\frac{f : X \twoheadrightarrow Y \qquad g_i : Y \rightharpoonup Z_i \qquad i = 1, 2}{f ; \langle\!\langle g_1, g_2 \rangle\!\rangle = \langle\!\langle f ; g_1, f ; g_2 \rangle\!\rangle : X \rightharpoonup Z_1 \times Z_2} \tag{8}$$

Proof. We will show now that the first rule is valid. The validity of the second rule is shown similarly. We have that

$$\begin{aligned}
\langle\!\langle f, g \rangle\!\rangle ; \varpi_1 &= \langle\!\langle f, g \rangle\!\rangle; P\varpi_1; \mu && [; \text{ def}]\\
&= \langle f, g \rangle; \psi; P(\pi_1; \eta); \mu && [\langle\!\langle \cdot, \cdot \rangle\!\rangle \text{ \& } \varpi_1 \text{ defs}]\\
&= \langle f, g \rangle; \psi; P\pi_1; P\eta; \mu && [P \text{ functor}]\\
&= \langle f, g \rangle; \psi; P\pi_1 && [P, \eta, \mu \text{ monad}]\\
&= \langle f, g \rangle; \pi_1 && [\psi \text{ properties}]\\
&= f.
\end{aligned}$$

For the third rule, we have that $h = h'; \eta = Hh' : X \to P(Y \times Z)$ for some $h' : X \to Y \times Z$. Since H is a functor and it also commutes with the pairing operation, we get that

$$\begin{aligned}
\langle\!\langle h ; \varpi_1, h ; \varpi_2 \rangle\!\rangle &= \langle\!\langle Hh' ; H\pi_1, Hh' ; H\pi_2 \rangle\!\rangle\\
&= H\langle h'; \pi_1, h'; \pi_2 \rangle,
\end{aligned}$$

which is equal to $Hh' = h$. For the fourth rule, we have that $f = f'; \eta = Hf' : X \to PY$ for some $f' : X \to Y$. Now,

$$\begin{aligned}
f\,;\langle\!\langle g_1, g_2\rangle\!\rangle &= f; P\langle\!\langle g_1, g_2\rangle\!\rangle; \mu && [\,;\ \text{def}]\\
&= f'; \eta; P\langle\!\langle g_1, g_2\rangle\!\rangle; \mu && [f \text{ deterministic}]\\
&= f'; \langle\!\langle g_1, g_2\rangle\!\rangle; \eta; \mu && [\eta \text{ natural}]\\
&= f'; \langle\!\langle g_1, g_2\rangle\!\rangle && [\text{monad}]\\
&= f'; \langle g_1, g_2\rangle; \psi && [\langle\!\langle\cdot,\cdot\rangle\!\rangle \text{ def}]\\
&= \langle f'; g_1, f'; g_2\rangle; \psi && [\mathcal{C} \text{ cartesian}]\\
&= \langle (f'; \eta)\,; g_1, (f'; \eta)\,; g_2\rangle; \psi && [\text{Remark 4}]\\
&= \langle\!\langle f\,; g_1, f\,; g_2\rangle\!\rangle,
\end{aligned}$$

and the proof is complete. □

Definition 9: We define the operation $\otimes$ on $\mathcal{C}_P$, which we call *Kleisli product functor*, as follows:

$$\frac{f_1 : X_1 \rightharpoonup Y_1 \qquad f_2 : X_2 \rightharpoonup Y_2}{f_1 \otimes f_2 := (f_1 \times f_2); \psi : X_1 \times X_2 \rightharpoonup Y_1 \times Y_2}.$$

Equivalently, we can define the Kleisli product as $f_1 \otimes f_2 = \langle\!\langle \varpi_1\,; f_1, \varpi_2\,; f_2\rangle\!\rangle$. Indeed,

$$\begin{aligned}
f_1 \otimes f_2 &= (f_1 \times f_2); \psi && [\otimes \text{ def}]\\
&= \langle \pi_1; f_1, \pi_2; f_2\rangle; \psi && [\times \text{ def}]\\
&= \langle (\pi_1; \eta)\,; f_1, (\pi_2; \eta)\,; f_2\rangle; \psi && [\text{Remark 4}]\\
&= \langle \varpi_1\,; f_1, \varpi_2\,; f_2\rangle; \psi && [\varpi \text{ def}]\\
&= \langle\!\langle \varpi_1\,; f_1, \varpi_2\,; f_2\rangle\!\rangle. && [\langle\!\langle\cdot,\cdot\rangle\!\rangle \text{ def}]
\end{aligned}$$

We observe the similarity in the definitions of the product functor $\times$ in $\mathcal{C}$ and the Kleisli product functor $\otimes$ in $\mathcal{C}_P$:

$$\frac{f_i : X_i \to Y_i \qquad i = 1, 2}{f_1 \times f_2 := \langle \pi_1; f_1, \pi_2; f_2\rangle} \qquad \frac{f_i : X_i \rightharpoonup Y_i \qquad i = 1, 2}{f_1 \otimes f_2 := \langle\!\langle \varpi_1; f_1, \varpi_2; f_2\rangle\!\rangle}$$

We have to see that $\otimes$ is indeed a functor. This is shown in Theorem 10 that follows. We also observe that H commutes with the product functor: $H(f \times g) = H\langle \pi_1; f, \pi_2; g\rangle = \langle\!\langle H(\pi_1; f), H(\pi_2; g)\rangle\!\rangle = \langle\!\langle \varpi_1; Hf, \varpi_2; Hg\rangle\!\rangle = Hf \otimes Hg$.

Theorem 10: The map $\otimes$ is a functor from the product category $\mathcal{C}_P \times \mathcal{C}_P$ to $\mathcal{C}_P$.

Proof. We first verify that $\eta_X \otimes \eta_Y = (\eta_X \times \eta_Y); \psi = \eta_{X \times Y} : X \times Y \rightharpoonup X \times Y$, using the ψ axioms. It remains to see that the rule

$$\frac{f_i : X_i \rightharpoonup Y_i \qquad g_i : Y_i \rightharpoonup Z_i \qquad i = 1, 2}{(f_1 \otimes f_2)\,; (g_1 \otimes g_2) = (f_1\,; g_1) \otimes (f_2\,; g_2)} \tag{9}$$

is valid. Indeed, we have that

$$\begin{aligned}
&(f_1 \otimes f_2)\,; (g_1 \otimes g_2) = \\
&(f_1 \times f_2); \psi; P[(g_1 \times g_2); \psi]; \mu = && [\,;\ \&\ \otimes \text{ defs}]\\
&(f_1 \times f_2); \psi; P(g_1 \times g_2); P\psi; \mu = && [P \text{ functor}]\\
&(f_1 \times f_2); (Pg_1 \times Pg_2); \psi; P\psi; \mu = && [\psi \text{ natural}]\\
&(f_1 \times f_2); (Pg_1 \times Pg_2); (\mu \times \mu); \psi = && [\psi \text{ axioms}]\\
&(f_1; Pg_1; \mu \times f_2; Pg_2; \mu); \psi && [\mathcal{C} \text{ cartesian}]
\end{aligned}$$

which is equal to $(f_1\,; g_1 \times f_2\,; g_2); \psi = f_1\,; g_1 \otimes f_2\,; g_2$. □

IV. Nondeterministic Monads

In this section we look at three equivalent ways of endowing with (angelic) nondeterministic structure the homsets of the Kleisli category of a monad. See also [15]. The proofs of equivalence we present make essential use of the cartesian structure of the base category.

Definition 11: Let (P, η, μ) be a monad over the category $\mathcal{C}$, $\perp_{XY} : X \to PY$ be a family of morphisms, and $+$ be a binary operation on $\mathcal{C}(X, PY)$

$$\frac{f : X \to PY \qquad g : X \to PY}{f + g : X \to PY},$$

which we call *(nondeterministic) choice*. We say that the $(P, \eta, \mu), +, \perp$ is a *nondeterministic monad* if the axioms

$$\begin{aligned}
(f + g) + h &= f + (g + h) & f\,; (g_1 + g_2) &= f\,; g_1 + f\,; g_2\\
f + g &= g + f & (f_1 + f_2)\,; g &= f_1\,; g + f_2\,; g\\
f + \perp &= f\\
f + f &= f
\end{aligned}$$

are satisfied. The above axioms state that every homset $\mathcal{C}(X, PY)$ is a commutative idempotent monoid w.r.t. $+$ and $\perp$, and that $+$ distributes over Kleisli composition.

Assuming the category $\mathcal{C}$ is cartesian, we will give an equivalent definition of the nondeterministic monad in terms of a natural transformation $\mathsf{u}_X : PX \times PX \to PX$, which we can intuitively think of as binary union. Then, we will also derive another equivalent definition of the nondeterministic monad in terms of a natural transformation $\mathsf{d}_X : X \times X \to PX$, which we think of as an operation that forms unordered pairs.

Theorem 12: Let $\mathcal{C}$ be a category with cartesian structure given by $\times, \pi_1, \pi_2, \langle\cdot,\cdot\rangle, \mathbb{1}, \perp$.

- Suppose $(P, \eta, \mu), +, \perp$ is a nondeterministic monad. Define $\mathsf{u}_X := \pi_1 + \pi_2 : PX \times PX \to PX$. Then, u is a natural transformation and the diagrams of Figure 4 commute.
- Conversely, suppose that (P, η, μ) is a monad, $\mathsf{u}_X : PX \times PX \to PX$ is a natural transformation, and $\perp_{\mathbb{1}X} : \mathbb{1} \to PX$ is a family of morphisms, so that the axioms of Figure 4 are satisfied. Define the operation $+$ by

$$\frac{f : X \to PY \qquad g : X \to PY}{f + g := \langle f, g\rangle; \mathsf{u}_Y : X \to PY}.$$

Also define the family of morphisms $\perp_{XY} := \perp_X; \perp_{\mathbb{1}Y} : X \to PY$. Then, $(P, \eta, \mu), +, \perp$ is a nondeterministic monad.

Proof. First, we will show a few simple consequences of $(P, \eta, \mu), +, \perp$ being a nondeterministic monad.

$$\frac{f : X \to Y \qquad g_i : Y \to PZ \qquad i = 1, 2}{f; (g_1 + g_2) = f; g_1 + f; g_2 : X \to PZ} \tag{10}$$

Using Remark 4 and the distributivity property, we have that $f; (g_1 + g_2) = (f; \eta)\,; (g_1 + g_2) = (f; \eta)\,; g_1 + (f; \eta)\,; g_2 = f; g_1 + f; g_2$.

$$\frac{f_i : X \to PY \qquad i = 1, 2}{P(f_1 + f_2); \mu = Pf_1; \mu + Pf_2; \mu : PX \to PY} \tag{11}$$

$$\begin{tikzcd}
(PX \times PX) \times PX \arrow[r, "\alpha"] \arrow[d, "\mathsf{u} \times \mathrm{id}"'] & PX \times (PX \times PX) \arrow[r, "\mathrm{id} \times \mathsf{u}"] & PX \times PX \arrow[d, "\mathsf{u}"] \\
PX \times PX \arrow[rr, "\mathsf{u}"] & & PX
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{1} \times PX \arrow[r, "\bot \times \mathrm{id}"] \arrow[dr, "\pi_2"'] & PX \times PX \arrow[d, "\mathsf{u}"] & PX \times \mathbb{1} \arrow[l, "\mathrm{id} \times \bot"'] \arrow[dl, "\pi_1"] \\
& PX &
\end{tikzcd}$$

$$\begin{tikzcd}
PX \times PX \arrow[r, "\mathsf{s}"] \arrow[dr, "\mathsf{u}"'] & PX \times PX \arrow[d, "\mathsf{u}"] \\
& PX
\end{tikzcd}
\qquad
\begin{tikzcd}
PX \arrow[r, "\langle \mathrm{id}, \mathrm{id} \rangle"] \arrow[dr, "\mathrm{id}"'] & PX \times PX \arrow[d, "\mathsf{u}"] \\
& PX
\end{tikzcd}$$

$$\begin{tikzcd}
P^2X \times P^2X \arrow[r, "\mathsf{u}_{PX}"] \arrow[d, "\mu \times \mu"'] & P^2X \arrow[d, "\mu"] \\
PX \times PX \arrow[r, "\mathsf{u}"] & PX
\end{tikzcd}
\qquad
\begin{tikzcd}
P(PX \times PX) \arrow[r, "P\mathsf{u}"] \arrow[d, "\langle P\pi_1, P\pi_2 \rangle"'] & P^2X \arrow[dd, "\mu_X"] \\
P^2X \times P^2X \arrow[d, "\mathsf{u}_{PX}"'] & \\
P^2X \arrow[r, "\mu_X"] & PX
\end{tikzcd}$$

Fig. 4. Commutative diagrams for $\mathsf{u}_X : PX \times PX \to PX$.

Using the distributivity property, we have that $P(f_1+f_2); \mu = \mathrm{id}; (f_1 + f_2) = \mathrm{id}; f_1 + \mathrm{id}; f_2 = Pf_1; \mu + Pf_2; \mu$.

$$\frac{f_i : X \to PY \qquad i = 1,2 \qquad g : Y \to Z}{(f_1 + f_2); Pg = f_1; Pg + f_2; Pg : X \to PZ} \tag{12}$$

As before, we have that $(f_1 + f_2); Pg = (f_1 + f_2); (g; \eta_Z) = f_1; (g; \eta_Z) + f_2; (g; \eta_Z) = f_1; Pg + f_2; Pg$.

$$\frac{f_i : X \to P^2Y \qquad i = 1,2}{(f_1 + f_2); \mu_Y = f_1; \mu_Y + f_2; \mu_Y : X \to PY} \tag{13}$$

We have that $(f_1+f_2); \mu = (f_1+f_2); \mathrm{id}_{PY} = f_1; \mathrm{id}+f_2; \mathrm{id} = f_1; \mu + f_2; \mu$.

Now, we will see that u is a natural transformation. For $f : X \to Y$, we claim that $\mathsf{u}_X; Pf = (Pf \times Pf); \mathsf{u} : PX \times PX \to PY$. Indeed, using rule (12) we see that $\mathsf{u}; Pf = (\pi_1 + \pi_2); Pf = \pi_1; Pf + \pi_2; Pf$, and using rule (10) that $(Pf \times Pf); \mathsf{u} = \langle \pi_1; Pf, \pi_2; Pf \rangle; (\pi_1 + \pi_2) = \pi_1; Pf + \pi_2; Pf$.

We will now show that the first four diagrams of Figure 4 commute. These diagrams say that for every object X, the triple $(X, \mathsf{u}_X, \bot_{\mathbb{1}X})$ is a commutative idempotent monoid in the cartesian category $\mathcal{C}$.

$$\begin{aligned}
\alpha; (\mathrm{id} \times \mathsf{u}); \mathsf{u} &= \langle \pi_1; \pi_1, \langle \pi_1; \pi_2, \pi_2 \rangle \rangle; (\mathrm{id} \times \mathsf{u}); \mathsf{u} \\
&= \langle \pi_1; \pi_1, \langle \pi_1; \pi_2, \pi_2 \rangle; \mathsf{u} \rangle; \mathsf{u} \\
&= \langle \pi_1; \pi_1, \langle \pi_1; \pi_2, \pi_2 \rangle; (\pi_1 + \pi_2) \rangle; \mathsf{u} \\
&= \langle \pi_1; \pi_1, \pi_1; \pi_2 + \pi_2 \rangle; (\pi_1 + \pi_2) \\
&= \pi_1; \pi_1 + (\pi_1; \pi_2 + \pi_2)
\end{aligned}$$

Also, $(\mathsf{u} \times \mathrm{id}); \mathsf{u} = \langle \pi_1; \mathsf{u}, \pi_2 \rangle; (\pi_1 + \pi_2) = \pi_1; \mathsf{u} + \pi_2 = \pi_1; (\pi_1+\pi_2)+\pi_2 = (\pi_1; \pi_1+\pi_1; \pi_2)+\pi_2$. From associativity of $+$ we obtain that $\alpha; (\mathrm{id} \times \mathsf{u}); \mathsf{u} = (\mathsf{u} \times \mathrm{id}); \mathsf{u}$.

For the second diagram (left and right identity) we have that $(\bot \times \mathrm{id}); \mathsf{u} = \langle \pi_1; \bot, \pi_2 \rangle; \mathsf{u} = \langle \bot, \pi_2 \rangle; (\pi_1 + \pi_2) = \bot + \pi_2 = \pi_2$ and similarly $(\mathrm{id} \times \bot); \mathsf{u} = \pi_1$. For commutativity and idempotence respectively we have that $\mathsf{s}; \mathsf{u} = \langle \pi_2, \pi_1 \rangle; (\pi_1 + \pi_2) = \pi_2 + \pi_1 = \pi_1 + \pi_2 = \mathsf{u}$ and $\langle \mathrm{id}, \mathrm{id} \rangle; \mathsf{u} = \langle \mathrm{id}, \mathrm{id} \rangle; (\pi_1 + \pi_1) = \mathrm{id} + \mathrm{id} = \mathrm{id}$. For the last two diagrams of Figure 4, we have that $(\mu \times \mu); \mathsf{u} = \langle \pi_1; \mu, \pi_2; \mu \rangle; (\pi_1+\pi_1) = \pi_1; \mu+\pi_2; \mu = (\pi_1+\pi_1); \mu = \mathsf{u}; \mu$ using rule (13), and also that $\langle P\pi_1, P\pi_2 \rangle; \mathsf{u}; \mu = (P\pi_1 + P\pi_2); \mu = P\pi_1; \mu + P\pi_2; \mu = P(\pi_1 + \pi_2); \mu = P\mathsf{u}; \mu$ using rule (13) and rule (11).

For the converse, we will first verify the properties stating that $\mathcal{C}(X, PY)$ is a commutative idempotent monoid w.r.t. $+$ and $\bot$. For associativity, we have that

$$\begin{aligned}
(f + g) + h &= \langle \langle f, g \rangle; \mathsf{u}, h \rangle; \mathsf{u} \\
&= \langle \langle f, g \rangle, h \rangle; (\mathsf{u} \times \mathrm{id}); \mathsf{u} \\
&= \langle \langle f, g \rangle, h \rangle; \alpha; (\mathrm{id} \times \mathsf{u}); \mathsf{u} \\
&= \langle f, \langle g, h \rangle \rangle; (\mathrm{id} \times \mathsf{u}); \mathsf{u} \\
&= \langle f, \langle g, h \rangle; \mathsf{u} \rangle; \mathsf{u} \\
&= f + (g + h).
\end{aligned}$$

We also have that $f + \bot = \langle f, \bot \rangle; \mathsf{u} = \langle f, \bot; \bot \rangle; \mathsf{u} = \langle f, \bot \rangle; (\mathrm{id} \times \bot); \mathsf{u} = \langle f, \bot \rangle; \pi_1 = f$ and similarly $\bot + f = f$. For commutativity and idempotence we have: $f + g = \langle f, g \rangle; \mathsf{u} = \langle f, g \rangle; \mathsf{s}; \mathsf{s}; \mathsf{u} = \langle g, f \rangle; \mathsf{u} = g + f$ and $f + f = \langle f, f \rangle; \mathsf{u} = f; \langle \mathrm{id}, \mathrm{id} \rangle; \mathsf{u} = f$. We show now the distributivity properties

$$\begin{aligned}
f; (g_1 + g_2) &= f; P(\langle g_1, g_2 \rangle; \mathsf{u}); \mu \\
&= f; P\langle g_1, g_2 \rangle; P\mathsf{u}; \mu \\
&= f; P\langle g_1, g_2 \rangle; \langle P\pi_1, P\pi_2 \rangle; \mathsf{u}; \mu \\
&= f; P\langle g_1, g_2 \rangle; \langle P\pi_1, P\pi_2 \rangle; (\mu \times \mu); \mathsf{u} \\
&= \langle f; P\langle g_1, g_2 \rangle; P\pi_1; \mu, f; P\langle g_1, g_2 \rangle; P\pi_2; \mu \rangle; \mathsf{u} \\
&= \langle f; Pg_1; \mu, f; Pg_2; \mu \rangle; \mathsf{u} \\
&= f; g_1 + f; g_2 \\
(f_1 + f_2); g &= \langle f_1, f_2 \rangle; \mathsf{u}; Pg; \mu \\
&= \langle f_1, f_2 \rangle; (Pg \times Pg); \mathsf{u}; \mu \\
&= \langle f_1, f_2 \rangle; (Pg \times Pg); (\mu \times \mu); \mathsf{u} \\
&= \langle f_1; Pg; \mu, f_2; Pg; \mu \rangle; \mathsf{u} \\
&= f_1; g + f_2; g
\end{aligned}$$

and the proof is complete. □

Theorem 13: Let $\mathcal{C}$ be a category with cartesian structure given by $\times, \pi_1, \pi_2, \langle \cdot, \cdot \rangle, \mathbb{1}, \bot$. Let (P, η, μ) be a monad over $\mathcal{C}$, and $\bot_{\mathbb{1}X} : \mathbb{1} \to PX$ be a family of morphisms.

- Suppose that $\mathsf{u}_X : PX \times PX \to PX$ is a natural transformation such that the diagrams of Figure 4 commute. Define $\mathsf{d}_X := (\eta_X \times \eta_X); \mathsf{u}_X : X \times X \to PX$. Then, d is a natural transformation satisfying the axioms of Figure 5.
- Conversely, suppose that $\mathsf{d}_X : X \times X \to PX$ is a natural transformation such that the diagrams of Figure 5 commute. Define $\mathsf{u}_X := \mathsf{d}_{PX}; \mu_X : PX \times PX \to PX$. Then, u is a natural transformation satisfying the axioms of Figure 4.

Proof. First, we will see that $\mathsf{d}_X : X \times X \to PX$ is a natural transformation. Using the facts that u and η are natural

$$\begin{tikzcd}
(X\times X)\times X \arrow[r, "\alpha"] \arrow[d, "\mathsf{d}\times\eta"'] & X\times(X\times X) \arrow[r, "\eta\times\mathsf{d}"] & PX\times PX \arrow[r, "d_{PX}"] & P^2X \arrow[d, "\mu"] \\
PX\times PX \arrow[rr, "d_{PX}"] & & P^2X \arrow[r, "\mu"] & PX
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{1}\times PX \arrow[r, "\perp\times\mathrm{id}"] \arrow[ddr, "\pi_2"'] & PX\times PX \arrow[d, "\mathsf{d}_{PX}"] & PX\times\mathbb{1} \arrow[l, "\mathrm{id}\times\perp"'] \arrow[ddl, "\pi_1"] \\
& P^2X \arrow[d, "\mu"] & \\
& PX &
\end{tikzcd}$$

$$\begin{tikzcd}
X\times X \arrow[r, "\mathsf{s}"] \arrow[dr, "\mathsf{d}"'] & X\times X \arrow[d, "\mathsf{d}"] & X \arrow[r, "\langle\mathrm{id},\mathrm{id}\rangle"] \arrow[dr, "\eta"'] & X\times X \arrow[d, "\mathsf{d}"] \\
& PX & & PX
\end{tikzcd}$$

$$\begin{tikzcd}
P^2X\times P^2X \arrow[r, "\mathsf{d}_{P^2X}"] \arrow[d, "\mu\times\mu"'] & P^3X \arrow[d, "\mu_{PX}"] & P(X\times X) \arrow[r, "P\mathsf{d}"] \arrow[d, "\langle P\pi_1, P\pi_2\rangle"'] & P^2X \arrow[dd, "\mu"] \\
PX\times PX \arrow[d, "\mathsf{d}_{PX}"'] & P^2X \arrow[d, "\mu"] & PX\times PX \arrow[d, "\mathsf{d}_{PX}"'] & \\
P^2X \arrow[r, "\mu"] & PX & P^2X \arrow[r, "\mu"] & PX
\end{tikzcd}$$

Fig. 5. Commutative diagrams for $\mathsf{d}_X : X\times X \to PX$.

transformations, we get that $\mathsf{d}; Pf = (\eta\times\eta); \mathsf{u}; Pf = (\eta\times\eta); (Pf\times Pf); \mathsf{u} = (\eta; Pf\times\eta; Pf); \mathsf{u} = (f;\eta\times f;\eta); \mathsf{u} = (f\times f); (\eta\times\eta); \mathsf{u} = (f\times f); \mathsf{d}$.

To verify that the first diagram commutes, we first notice that $\mathsf{d}_{PX}; \mu = (\eta_{PX}\times\eta_{PX}); \mathsf{u}_{PX}; \mu = (\eta_{PX}\times\eta_{PX}); (\mu\times\mu); \mathsf{u} = (\eta_{PX};\mu\times\eta_{PX};\mu); \mathsf{u} = \mathsf{u}$. Then, we have that $\alpha; (\eta\times\mathsf{d}); \mathsf{d}_{PX}; \mu = \alpha; (\eta\times(\eta\times\eta);\mathsf{u}); \mathsf{u} = \alpha; (\eta\times(\eta\times\eta)); (\mathrm{id}\times\mathsf{u}); \mathsf{u} = ((\eta\times\eta)\times\eta); \alpha; (\mathrm{id}\times\mathsf{u}); \mathsf{u}$, and $(\mathsf{d}\times\eta); \mathsf{d}_{PX}; \mu = ((\eta\times\eta);\mathsf{u}\times\eta); \mathsf{u} = ((\eta\times\eta)\times\eta); (\mathsf{u}\times\mathrm{id}); \mathsf{u}$. Since $\alpha; (\mathrm{id}\times\mathsf{u}); \mathsf{u} = (\mathsf{u}\times\mathrm{id}); \mathsf{u}$, the diagram commutes.

It is immediate that the second diagram commutes: $(\perp\times\mathrm{id}); \mathsf{d}_{PX}; \mu = (\perp\times\mathrm{id}); \mathsf{u} = \pi_2$ and similarly $(\mathrm{id}\times\perp); \mathsf{d}_{PX}; \mu = (\mathrm{id}\times\perp); \mathsf{u} = \pi_1$. For the next three diagrams we have: $\mathsf{s}; \mathsf{d} = \mathsf{s}; (\eta\times\eta); \mathsf{u} = (\eta\times\eta); \mathsf{s}; \mathsf{u} = (\eta\times\eta); \mathsf{u} = \mathsf{d}$, $\langle\mathrm{id},\mathrm{id}\rangle; \mathsf{d} = \langle\mathrm{id},\mathrm{id}\rangle; (\eta\times\eta); \mathsf{u} = \eta; \langle\mathrm{id},\mathrm{id}\rangle; \mathsf{u} = \eta$, and $\mathsf{d}_{P^2X}; \mu_{PX}; \mu = \mathsf{u}_{PX}; \mu = (\mu\times\mu); \mathsf{u} = (\mu\times\mu); \mathsf{d}_{PX}; \mu$ respectively. Finally,

$$\begin{aligned}
&\langle P\pi_1, P\pi_2\rangle; \mathsf{d}_{PX}; \mu = \\
&\langle P\pi_1, P\pi_2\rangle; (\eta_{PX}\times\eta_{PX}); \mathsf{u}_{PX}; \mu = \\
&\langle P\pi_1, P\pi_2\rangle; (\eta_{PX}\times\eta_{PX}); (\mu\times\mu); \mathsf{u} = \\
&\langle P\pi_1, P\pi_2\rangle; (P\eta;\mu\times P\eta;\mu); \mathsf{u} = \\
&\langle P\pi_1; P\eta, P\pi_2; P\eta\rangle; (\mu\times\mu); \mathsf{u} = \\
&\langle P((\eta\times\eta);\pi_1), P((\eta\times\eta);\pi_2)\rangle; (\mu\times\mu); \mathsf{u} = \\
&P(\eta\times\eta); \langle P\pi_1, P\pi_2\rangle; \mathsf{u}_{PX}; \mu,
\end{aligned}$$

which is equal to $P(\eta\times\eta); P\mathsf{u}; \mu = P\mathsf{d}; \mu$. So, the last diagram commutes.

For the converse, we will first show that $\mathsf{u} : PX\times PX \to PX$ is a natural transformation. Since d and μ are natural transformations, we have that $(Pf\times Pf); \mathsf{u}_Y = (Pf\times Pf); \mathsf{d}_{PY}; \mu_Y = \mathsf{d}_{PX}; P^2f; \mu_Y = \mathsf{d}_{PX}; \mu_X; Pf = \mathsf{u}_X; Pf$. For the first diagram we see that

$$\begin{aligned}
\alpha; (\mathrm{id}\times\mathsf{u}); \mathsf{u} &= \alpha; (\eta;\mu\times\mathsf{d};\mu); \mathsf{d}; \mu \\
&= \alpha; (\eta\times\mathsf{d}); (\mu\times\mu); \mathsf{d}; \mu \\
&= \alpha; (\eta\times\mathsf{d}); \mathsf{d}; \mu; \mu \\
&= (\mathsf{d}\times\eta); \mathsf{d}; \mu; \mu \\
&= (\mathsf{d}\times\eta); (\mu\times\mu); \mathsf{d}; \mu \\
&= (\mathsf{d};\mu\times\eta;\mu); \mathsf{d}; \mu \\
&= (\mathsf{u}\times\mathrm{id}); \mathsf{u}
\end{aligned}$$

and therefore it commutes. Showing that the rest of the diagrams commute is straightforward. □

V. Nondeterministic Strong Monad with Iteration

In Section IV we investigated the nondeterministic structure of the Kleisli category of a monad in isolation from products. This is not sufficient for our purposes, because we also want to capture the interaction between nondeterminism and products. One step towards this direction is made by considering an additional axiom that relates the tensorial strength with the nondeterministic structure (Theorem 14).

Theorem 14: Let $\mathcal{C}$ be a category with cartesian structure given by $\times, \pi_1, \pi_2, \langle\cdot,\cdot\rangle, \mathbb{1}, \perp$. Let $(P,\eta,\mu), t$ be a commutative strong monad over $\mathcal{C}$ with tensorial strength $t_{X,Y} : X\times PY \to P(X\times Y)$. Assume additionally that (P,η,μ) is a nondeterministic monad together with $+$ and $\perp$, and also that the axiom

$$\frac{f : X\to PY \qquad g : X \to PY}{\langle\mathrm{id}, f+g\rangle; t = \langle\mathrm{id}, f\rangle; t + \langle\mathrm{id}, g\rangle; t : X \to P(X\times Y)}.$$

holds. Then, the rules

$$\frac{f_i : X\to PY \qquad i=1,2 \qquad g : X\to PZ}{\langle\!\langle f_1+f_2, g\rangle\!\rangle = \langle\!\langle f_1, g\rangle\!\rangle + \langle\!\langle f_2, g\rangle\!\rangle : X \to P(Y\times Z)} \tag{14}$$

$$\frac{f : X\to PY \qquad g_i : X\to PZ \qquad i=1,2}{\langle\!\langle f, g_1+g_2\rangle\!\rangle = \langle\!\langle f, g_1\rangle\!\rangle + \langle\!\langle f, g_2\rangle\!\rangle : X \to P(Y\times Z)} \tag{15}$$

are valid.

Proof. First, we will show that the axiom we assumed can be generalized into the rule

$$\frac{f : X\to Y \qquad g_i : X\to PZ \qquad i=1,2}{\langle f, g_1+g_2\rangle; t = \langle f, g_1\rangle; t + \langle f, g_2\rangle; t : X \to P(Y\times Z)}.$$

Since t is a natural transformation, we have that

$$\begin{aligned}
\langle f, g_1+g_2\rangle; t &= \langle\mathrm{id}, g_1+g_2\rangle; (f\times P\mathrm{id}); t \\
&= \langle\mathrm{id}, g_1+g_2\rangle; t; P(f\times\mathrm{id}) \\
&= (\langle\mathrm{id}, g_1\rangle; t + \langle\mathrm{id}, g_2\rangle; t); P(f\times\mathrm{id}) \\
&= \langle\mathrm{id}, g_1\rangle; t; P(f\times\mathrm{id}) + \langle\mathrm{id}, g_2\rangle; t; P(f\times\mathrm{id}) \\
&= \langle f, g_1\rangle; t + \langle f, g_2\rangle; t.
\end{aligned}$$

Now, it is easy to show that the dual rule

$$\frac{f_i : X\to PY \qquad i=1,2 \qquad g : X\to Z}{\langle f_1+f_2, g\rangle; \tau = \langle f_1, g\rangle; \tau + \langle f_2, g\rangle; \tau : X \to P(Y\times Z)}$$

holds for the dual tensorial strength $\tau_{X,Y} := \mathsf{s}; t; P\mathsf{s} : PX \times Y \to P(X \times Y)$. So, for rule (14) we have that

$$\begin{aligned}\langle\!\langle f_1 + f_2, g\rangle\!\rangle &= \langle f_1 + f_2, g\rangle; \tau; Pt; \mu \\ &= (\langle f_1, g\rangle; \tau + \langle f_2, g\rangle; \tau); Pt; \mu \\ &= \langle f_1, g\rangle; \tau; Pt; \mu + \langle f_2, g\rangle; \tau; Pt; \mu \\ &= \langle\!\langle f_1, g\rangle\!\rangle + \langle\!\langle f_2, g\rangle\!\rangle\end{aligned}$$

and, similarly,

$$\begin{aligned}\langle\!\langle f, g_1 + g_2\rangle\!\rangle &= \langle f, g_1 + g_2\rangle; t; P\tau; \mu \\ &= (\langle f, g_1\rangle; t + \langle f, g_2\rangle; t); P\tau; \mu \\ &= \langle f, g_1\rangle; t; P\tau; \mu + \langle f, g_2\rangle; t; P\tau; \mu \\ &= \langle\!\langle f, g_1\rangle\!\rangle + \langle\!\langle f, g_2\rangle\!\rangle.\end{aligned}$$

for rule (15). $\square$

Definition 15: Let $\mathcal{C}$ be a category with cartesian structure and bottom elements given by $\times, \pi_1, \pi_2, \langle\cdot,\cdot\rangle, \mathbb{1}, \perp$. We say that $(P, \eta, \mu), \psi, \mathsf{u},^*$ is a *nondeterministic strong monad with iteration* is the following are satisfied:

(i) $P\mathbb{1} \cong \mathbb{1}$ (hence $\mathbb{1}$ is a terminal object of $\mathcal{C}_P$)

(ii) $(P, \eta, \mu), \psi$ is a commutative strong monad with lazy pairs.

(iii) $(P, \eta, \mu), \mathsf{u}, \bot$ is a nondeterministic monad, where $\bot_{XY} = \perp_{XY}; \eta_Y : X \to PY$.

(iv) For all $f, g : X \to PY$, the equation $\langle \mathrm{id}, f + g\rangle; t = \langle \mathrm{id}, f\rangle; t + \langle \mathrm{id}, g\rangle; t : X \to P(X \times Y)$ holds, where t is the tensorial strength induced by ψ and $+$ is the choice operation induced by u.

(v) The axiom $\mathrm{id}_{P(X \times Y)} \leq \langle P\pi_1, P\pi_2\rangle; \psi_{X,Y}$ holds, where $\leq$ is the partial order induced by $+$.

(vi) The *iteration* operation * sends a morphism $f : X \to PX$ to $f^* : X \to PX$. The axioms $\eta + f; f^* \leq f^*$, $\eta + f^*; f \leq f^*$, $f; g \leq g \Rightarrow f^*; g \leq g$, and $g; f \leq g \Rightarrow g; f^* \leq g$ are satisfied.

Theorem 16: Let $\mathcal{C}$ be a category with cartesian structure and bottom elements given by $\times, \pi_1, \pi_2, \langle\cdot,\cdot\rangle, \mathbb{1}, \perp$. Let $(P, \eta, \mu), \psi, \mathsf{u},^*$ be a nondeterministic strong monad with iteration. Then, the Kleisli category $\mathcal{C}_P$ with Kleisli composition $;$ and Kleisli identity η, together with the operations $\varpi_1, \varpi_2, \langle\!\langle\cdot,\cdot\rangle\!\rangle, \bot, +,^*$ (Kleisli projections, pairing, bottoms, nondeterministic choice, and iteration) satisfies the axioms of Table I.

Proof. The axioms of the *first group* follow from the definition of determinism and the fact that (P, η, μ) is a monad. The axioms of the *second group* follow from the fact that the strong monad $(P, \eta, \mu), \psi$ is commutative and satisfies the "lazy pairs" property (Theorem 8 and Theorem 10). The axioms of the *third group* follow from the requirement that $P\mathbb{1} \cong \mathbb{1}$ and from the definition $\bot_{XY} = H\perp_{XY}$, where H is the unit functor of the monad (recall that $\perp_{\mathbb{1}X}$ is a bottom global element). For the *fourth group* of axioms: Some of them are a restatement of the definition of $(P, \eta, \mu), +, \bot$ being a nondeterministic monad (see also Theorem 12). The last two of them follow from the requirement that $\langle \mathrm{id}, f + g\rangle; t =$

Kleisli identity $\eta_X : X \rightarrowtriangle X$, and Kleisli composition $;$.

$$\frac{f : X \rightharpoonup Y \qquad g : Y \rightharpoonup Z}{f; g : X \rightharpoonup Z} \qquad \frac{f : X \rightarrowtriangle Y \qquad g : Y \rightarrowtriangle Z}{f; g : X \rightarrowtriangle Z}$$

$$\frac{f : X \rightharpoonup Y}{\eta_X; f = f : X \rightharpoonup Y} \qquad \frac{f : X \rightharpoonup Y}{f; \eta_Y = f : X \rightharpoonup Y}$$

$$\frac{f : X \rightharpoonup Y \qquad g : Y \rightharpoonup Z \qquad h : Z \rightharpoonup W}{(f; g); h = f; (g; h) : X \rightharpoonup W}$$

Kleisli projections $\varpi_1 : X \times Y \rightarrowtriangle X$, $\varpi_2 : X \times Y \rightarrowtriangle Y$, Kleisli pairing $\langle\!\langle\cdot,\cdot\rangle\!\rangle$, Kleisli product functor $f \otimes g = \langle\!\langle f; \varpi_1, g; \varpi_2\rangle\!\rangle$.

$$\frac{f : X \rightharpoonup Z \qquad g : Y \rightharpoonup Z}{\langle\!\langle f, g\rangle\!\rangle : X \rightharpoonup Y \times Z} \qquad \frac{f : X \rightarrowtriangle Z \qquad g : Y \rightarrowtriangle Z}{\langle\!\langle f, g\rangle\!\rangle : X \rightarrowtriangle Y \times Z}$$

$$\frac{f : X \rightharpoonup Y \qquad g : X \rightharpoonup Z}{\langle\!\langle f, g\rangle\!\rangle; \varpi_1 = f : X \rightharpoonup Y} \qquad \frac{f : X \rightharpoonup Y \qquad g : X \rightharpoonup Z}{\langle\!\langle f, g\rangle\!\rangle; \varpi_2 = g : X \rightharpoonup Z}$$

$$\frac{h : X \rightarrowtriangle Y \times Z}{\langle\!\langle h; \varpi_1, h; \varpi_2\rangle\!\rangle = h : X \rightarrowtriangle Y \times Z}$$

$$\frac{f : X \rightarrowtriangle Y \qquad g_i : Y \rightharpoonup Z_i \qquad i = 1, 2}{f; \langle\!\langle g_1, g_2\rangle\!\rangle = \langle\!\langle f; g_1, f; g_2\rangle\!\rangle : X \rightharpoonup Z_1 \times Z_2}$$

$$\frac{f_i : X_i \rightharpoonup Y_i \qquad g_i : Y_i \rightharpoonup Z_i \qquad i = 1, 2}{(f_1 \otimes f_2); (g_1 \otimes g_2) = (f_1; g_1) \otimes (f_2; g_2)}$$

Kleisli bottom morphisms $\bot_{XY} : X \rightarrowtriangle Y$.

$$\frac{f : X \rightharpoonup \mathbb{1}}{f = \bot : X \rightharpoonup \mathbb{1}} \qquad \frac{f : X \rightharpoonup Y}{f; \bot = \bot : X \rightharpoonup Z}$$

Nondeterministic choice operation $+$. Partial order $\leq$ defined as: $f \leq g$ iff $f + g = g$, for $f, g : X \rightharpoonup Y$.

$$\frac{f : X \rightharpoonup Y \qquad g : X \rightharpoonup Y}{f + g : X \rightharpoonup Y}$$

$$\frac{f, g, h : X \rightharpoonup Y}{(f + g) + h = f + (g + h) : X \rightharpoonup Y} \qquad \frac{f, g : X \rightharpoonup Y}{f + g = g + f : X \rightharpoonup Y}$$

$$\frac{f : X \rightharpoonup Y}{f + \bot = f : X \rightharpoonup Y} \qquad \frac{f : X \rightharpoonup Y}{f + f = f : X \rightharpoonup Y}$$

$$\frac{f : X \rightharpoonup Y \qquad g_1, g_2 : Y \rightharpoonup Z}{f; (g_1 + g_2) = f; g_1 + f; g_2 : X \rightharpoonup Z}$$

$$\frac{f_1, f_2 : X \rightharpoonup Y \qquad g : Y \rightharpoonup Z}{(f_1 + f_2); g = f_1; g + f_2; g : X \rightharpoonup Z}$$

$$\frac{h : X \rightarrowtriangle Y \times Z}{h \leq \langle\!\langle h; \varpi_1, h; \varpi_2\rangle\!\rangle = h : X \rightarrowtriangle Y \times Z}$$

$$\frac{f_1, f_2 : X \rightharpoonup Y \qquad g : X \rightharpoonup Z}{\langle\!\langle f_1 + f_2, g\rangle\!\rangle = \langle\!\langle f_1, g\rangle\!\rangle + \langle\!\langle f_2, g\rangle\!\rangle : X \rightharpoonup Y \times Z}$$

$$\frac{f : X \rightharpoonup Y \qquad g_1, g_2 : X \rightharpoonup Z}{\langle\!\langle f, g_1 + g_2\rangle\!\rangle = \langle\!\langle f, g_1\rangle\!\rangle + \langle\!\langle f, g_2\rangle\!\rangle : X \rightharpoonup Y \times Z}$$

Iteration operation *.

$$\frac{f : X \rightharpoonup X}{f^* : X \rightharpoonup X}$$

$$\frac{f : X \rightharpoonup X}{\eta_X + f; f^* \leq f^* : X \rightharpoonup X} \qquad \frac{f : X \rightharpoonup X \qquad g : X \rightharpoonup Y}{f; g \leq g \Rightarrow f^*; g \leq g : X \rightharpoonup Y}$$

$$\frac{f : X \rightharpoonup X}{\eta_X + f^*; f \leq f^* : X \rightharpoonup X} \qquad \frac{g : X \rightharpoonup Y \qquad f : Y \rightharpoonup Y}{g; f \leq g \Rightarrow g; f^* \leq g : X \rightharpoonup Y}$$

TABLE I

KLEISLI CATEGORY OF A NONDETERMINISTIC STRONG MONAD WITH ITERATION.

$\langle \mathrm{id}, f\rangle; t + \langle \mathrm{id}, g\rangle; t$ as shown in Theorem 14. Finally, $h \leq \langle\!\langle h; \varpi_1, h; \varpi_2 \rangle\!\rangle$ follows from $\mathrm{id} \leq \langle P\pi_1, P\pi_2\rangle; \psi$, because it holds that $h; \varpi_i = h; P(\pi_i; \eta); \mu = h; P\pi_i; P\eta; \mu = h; P\pi_i$ and therefore $\langle\!\langle h; \varpi_1, h; \varpi_2\rangle\!\rangle = \langle\!\langle h; P\pi_1, h; P\pi_2\rangle\!\rangle = \langle h; P\pi_1, h; P\pi_2\rangle; \psi = h; \langle P\pi_1, P\pi_2\rangle; \psi \geq h$. The *fifth group* of axioms is an assumption we have made. □

A. The Lowerset Monad

In the usual relational interpretation of programs, a (strict) nondeterministic program $f : X \rightharpoonup Y$ is interpreted as a morphism in the category **Rel** of sets and binary relations. The category **Rel** is isomorphic to the Kleisli category $\mathbf{Set}_\wp$ of the powerset monad $\wp$ over the category **Set** of sets and total functions.

In order to deal explicitly with partiality we introduce a symbol $\perp$ that denotes divergence. We note that **Set** is isomorphic to the category **FlatST** of flat posets and strict total functions. By "total" we mean here that a morphism $f_\perp : X_\perp \to Y_\perp$ of **FlatST** sends every $x \neq \perp$ to $f_\perp(x) \neq \perp$. The isomorphism is witnessed by the *lifting functor* $(\cdot)_\perp$, defined as follows:

$$X \text{ in } \mathbf{Set} \mapsto X_\perp \text{ in } \mathbf{FlatST}$$
$$f : X \to Y \text{ in } \mathbf{Set} \mapsto f_\perp : X_\perp \to Y_\perp \text{ in } \mathbf{FlatST}$$

where $X_\perp := X \cup \{\perp_X\}$ together with the flat order $\leq$, $f_\perp(\perp_X) = \perp_Y$ and $f_\perp(x) = f(x)$ for $x \in X$. Define the monad $\wp_\perp$ over **FlatST** by

$$\wp_\perp(X_\perp) = \{\text{non-empty lower sets of } X_\perp\}$$
$$\wp_\perp(f_\perp : X_\perp \to Y_\perp) = \lambda S \in \wp_\perp(X_\perp).\{f_\perp(x) \mid x \in S\}$$

The bottom element of $\wp_\perp(X_\perp)$ is the singleton $\{\perp_X\}$. The unit of the monad is given by $x \in X_\perp \mapsto \{x, \perp_X\}$ and the multiplication is union. Under these definitions, the Kleisli category $\mathbf{Set}_\wp$ is isomorphic to $\mathbf{FlatST}_{\wp_\perp}$. The isomorphism is witnessed by the *(nondeterministic) lifting functor* given by:

$$X \text{ in } \mathbf{Set}_\wp \mapsto X_\perp \text{ in } \mathbf{FlatST}_{\wp_\perp}$$
$$f : X \to \wp(Y) \text{ in } \mathbf{Set}_\wp \mapsto \phi : X_\perp \to \wp_\perp(Y_\perp) \text{ in } \mathbf{FlatST}_{\wp_\perp}$$

where $\phi(\perp_X) = \{\perp_Y\}$ and $\phi(x) = f(x) \cup \{\perp_Y\}$ for $x \in X$.

Remark 17: We observe that in **FlatST**, the natural operation of forming *eager pairs* of elements does not give rise to a categorical product. Define the eager pair $\langle x, y\rangle$, where $x \in X_\perp$ and $y \in Y_\perp$, to be equal to $\perp_{X\times Y}$ if one of x, y is bottom and equal to the pair (x, y) otherwise. The *smash product* is defined as $X_\perp \otimes Y_\perp = (X \times Y)_\perp$. Now $\langle \perp, y\rangle; \pi_2 = \perp; \pi_2 = \perp$. Therefore, we cannot recover the right component $y \neq \perp$ of the pair. Thus the smash product is not categorical.

Consider now the category **Pposet** of *pointed posets* (posets with a bottom element) and monotone functions, in which we can interpret non-strict deterministic programs that form lazy pairs. The morphisms in **Pposet** can be partial, in the sense that they are allowed to send non-bottom elements to bottom. For an object $(X, \leq)$ of **Pposet**, we understand the partial order $\leq$ as follows: $x \leq y$ intuitively means that x "has more diverging components" than y.

Remark 18: **Pposet** is a cartesian category with bottom elements. The product $X \times Y$ of two objects X, Y is the cartesian product together with the pointwise partial order. So, the bottom element of $X \times Y$ is $\perp_{X\times Y} = \langle \perp_X, \perp_Y\rangle$. The projections morphisms π_1 and π_2 are given by $\pi_i(x_1, x_2) = x_i$. The pairing operation $\langle \cdot, \cdot \rangle$ is defined as

$$\frac{f : X \to Y \qquad g : X \to Z}{\langle f, g\rangle = \lambda x \in X.\langle f(x), g(x)\rangle}.$$

The terminal object is some singleton poset $\mathbb{1} = \{\perp_\mathbb{1}\}$. The bottom global element $\perp_{\mathbb{1}X} : \mathbb{1} \to X$ sends $\perp_\mathbb{1}$ to the bottom $\perp_X$ of X. So, $\perp_{XY} = \perp_X; \perp_{\mathbb{1}Y}$ is the function that always diverges. We define the pointwise partial order $\leq$ on every homset $\mathbf{Pposet}(X, Y)$. Then, $\perp_{XY}$ is the bottom element of $\mathbf{Pposet}(X, Y)$.

Definition 19 (lowerset monad): Let $(X, \leq)$ be a pointed poset, the bottom element of which is denoted $\perp_X$. For a subset $S \subseteq X$ define $\downarrow S = \{y \in X \mid y \leq x \text{ for some } x \in S\}$ to be the lowerset generated by S. For $x \in X$, we write $\downarrow x$ to mean $\downarrow\{x\}$. Define $\wp_\downarrow X$ to be the set of all non-empty lowersets of X. We observe that $\wp_\downarrow X$ is a complete lattice w.r.t. set inclusion. The top element is X and the bottom is $\{\perp_X\}$. The join is set-theoretic union and the meet is set-theoretic intersection. It follows that the homset $\mathbf{Pposet}(X, \wp_\downarrow Y)$ (w.r.t. the pointwise order induced by $\wp_\downarrow Y, \subseteq$) is also a complete lattice. We extend $\wp_\downarrow$ into an endofunctor $\mathbf{Pposet} \to \mathbf{Pposet}$ by putting

$$\frac{f : X \to Y \text{ in } \mathbf{Pposet}}{\wp_\downarrow f := \lambda S \in \wp_\downarrow X.\ \downarrow\{f(x) \mid x \in S\}}.$$

Together with the families of maps

$$\eta_X : x \in X \mapsto \downarrow x \in \wp_\downarrow X$$
$$\mu_X : \mathcal{S} \in \wp_\downarrow^2 X \mapsto \bigcup \mathcal{S} \in \wp_\downarrow X$$

it forms a monad over **Pposet**. We call $(\wp_\downarrow, \eta, \mu)$ the *lowerset monad* over **Pposet**. It can be easily shown that

$$\frac{f : X \to \wp_\downarrow Y \qquad g : Y \to \wp_\downarrow Z}{(f ; g)(x) = \bigcup_{y \in f(x)} g(y)}$$

gives Kleisli composition.

Theorem 20: Define the family of morphisms $\psi_{X,Y}$ as

$$\psi_{X,Y} : \wp_\downarrow X \times \wp_\downarrow Y \to \wp_\downarrow(X \times Y)$$
$$(S_1, S_2) \mapsto S_1 \times S_2$$

and the family of morphisms u_X as

$$\mathsf{u}_X : \wp_\downarrow X \times \wp_\downarrow X \to \wp_\downarrow X$$
$$(S_1, S_2) \mapsto S_1 \cup S_2$$

For a morphism $f : X \to \wp_\downarrow X$, define $f^i : X \to \wp_\downarrow X$, $i < \omega$, by induction: $f^0 = \eta_X$ and $f^{i+1} = f^i ; f$. Now, define the iteration operation by

$$f^* := \bigvee_{i<\omega} f^i = \lambda x \in X.\ \bigcup_{i<\omega} f^i(x)$$

where $\bigvee$ is the join of the complete lattice $\mathbf{Pposet}(X, \wp_\downarrow X)$. Then, the lowerset monad $(\wp_\downarrow, \eta, \mu)$ over the category **Pposet**,

$$\begin{tikzcd}
(S_1, S_2) \arrow[r, mapsto, "\wp_{\downarrow} f_1 \times \wp_{\downarrow} f_2"] \arrow[d, mapsto, "\psi"'] & (\downarrow f_1(S_1), \downarrow f_2(S_2)) \arrow[d, mapsto, "\psi"] \\
S_1 \times S_2 \arrow[r, mapsto, "\wp_{\downarrow}(f_1 \times f_2)"] & \downarrow(f_1(S_1) \times f_2(S_2))
\end{tikzcd}$$

$$\begin{tikzcd}
\left(\{S_i\}_{i\in I}, \{T_j\}_{j\in J}\right) \arrow[r, mapsto, "\mu \times \mu"] \arrow[d, mapsto, "\psi"'] & \left(\bigcup_{i\in I} S_i, \bigcup_{j\in J} T_j\right) \arrow[dd, mapsto, "\psi"] \\
\{(S_i, T_j)\}_{(i,j)\in I\times J} \arrow[d, mapsto, "\wp_{\downarrow}(\psi)"'] & \\
\{S_i \times T_j\}_{(i,j)\in I\times J} \arrow[r, mapsto, "\mu"] & \bigcup_{(i,j)\in I\times J} S_i \times T_j
\end{tikzcd}$$

Fig. 6. ψ is natural, and $\psi; \wp_{\downarrow}(\psi); \mu = (\mu \times \mu); \psi$.

together with the operations $\psi, \mathsf{u}, ^*$, is a nondeterministic strong monad with iteration.

Proof. First, we see that $\psi_{X,Y}$ is well-defined. This is the case, because if S_1, S_2 are non-empty lowersets of X, Y respectively, then $S_1 \times S_2$ is a non-empty lowerset of $X \times Y$. That ψ is a natural transformation is a consequence of the fact that $\downarrow A \times \downarrow B = \downarrow(A \times B)$, as can be seen in Figure 6. It is straightforward to show that ψ satisfies the properties of Figure 3. For example, we show in Figure 6 that the fourth diagram of Figure 3 commutes. We are making use of the simple fact that $(\bigcup_{i\in I} A_i) \times (\bigcup_{j\in J} B_j) = \bigcup_{(i,j)\in I\times J} A_i \times B_j$ for any collections of sets $\{A_i\}_{i\in I}$ and $\{B_j\}_{j\in J}$. We have thus established that $(\wp_{\downarrow}, \eta, \mu), \psi$ is a commutative strong monad with lazy pairs.

Now, we see that u_X is well-defined, because $S_1 \cup S_2$ is a lowerset whenever S_1 and S_2 are lowersets. We also define $\bot_{XY} = \bot_{XY}; \eta_Y$. So, $\bot_{XY}(x) = \{\bot_Y\}$ for all $x \in X$. That u is a natural transformation is a consequence of the fact that $\downarrow f(S) \cup \downarrow f(T) = \downarrow(f(S) \cup f(T)) = \downarrow f(S \cup T)$, as can be seen in Figure 7. The properties of Figure 4 hold. In Figure 7 we show that last two diagrams of Figure 4 commute. We are making use of the simple properties: $\bigcup(\{A_i\}_i \cup \{B_j\}_j) = (\bigcup_i A_i) \cup (\bigcup_j B_j)$ and $(\bigcup_i A_i) \cup (\bigcup_i B_i) = \bigcup_i (A_i \cup B_i)$. So, $(\wp_{\downarrow}, \eta, \mu), \mathsf{u}, \bot$ is a nondeterministic monad. The operation $+$ induced by u is given by $f + g = \lambda x \in X. f(x) \cup g(x)$ for $f, g : X \to \wp_{\downarrow} Y$.

The tensorial strength $t_{X,Y} : X \times \wp_{\downarrow} Y \to \wp_{\downarrow}(X \times Y)$ induced by ψ is given by $t_{X,Y} = (\eta \times \mathrm{id}); \psi = \lambda(x, S). \downarrow x \times S$. We verify the axiom $\langle \mathrm{id}, f + g\rangle; t = \langle \mathrm{id}, f\rangle; t + \langle \mathrm{id}, g\rangle; t$ for $f, g : X \to \wp_{\downarrow} Y$. This simply amounts to showing that $\downarrow x \times (f(x) \cup g(x)) = \downarrow x \times f(x) \cup \downarrow x \times g(x)$ for all $x \in X$, which is true. Moreover, we notice that for a lowerset $\{(x_i, y_i)\}_{i\in I}$ in $\wp_{\downarrow}(X \times Y)$ we have: $\{(x_i, y_i)\}_{i\in I} \mapsto (\{x_i\}_{i\in I}, \{y_i\}_{i\in I}) \mapsto \{(x_i, y_j)\}_{(i,j)\in I\times I}$ through $\langle \wp_{\downarrow}\pi_1, \wp_{\downarrow}\pi_2\rangle$ and ψ respectively. Since the set $\{(x_i, y_i)\}_{i\in I}$ is contained in $\{(x_i, y_j)\}_{(i,j)\in I\times I}$ we conclude that the property $\mathrm{id} \le \langle \wp_{\downarrow}\pi_1, \wp_{\downarrow}\pi_2\rangle; \psi : \wp_{\downarrow}(X \times Y) \to \wp_{\downarrow}(X \times Y)$ holds.

An easy induction gives us that $f^{i+1} = f; f^i$ for every $i < \omega$ (making use of the associativity property of Kleisli

$$\begin{tikzcd}
(S, T) \arrow[r, mapsto, "\wp_{\downarrow} f \times \wp_{\downarrow} f"] \arrow[d, mapsto, "\mathsf{u}"'] & (\downarrow f(S), \downarrow f(T)) \arrow[d, mapsto, "\mathsf{u}"] \\
S \cup T \arrow[r, mapsto, "\wp_{\downarrow} f"] & \downarrow f(S \cup T)
\end{tikzcd}$$

$$\begin{tikzcd}
(\{S_i\}_i, \{T_j\}_j) \arrow[r, mapsto, "\mathsf{u}"] \arrow[d, mapsto, "\mu \times \mu"'] & \{S_i\}_i \cup \{T_j\}_j \arrow[d, mapsto, "\mu"] \\
(\bigcup_i S_i, \bigcup_j T_j) \arrow[r, mapsto, "\mathsf{u}"] & (\bigcup_i S_i) \cup (\bigcup_j T_j)
\end{tikzcd}$$

$$\begin{tikzcd}
\{(S_i, T_i)\}_i \arrow[r, mapsto, "\langle \wp_{\downarrow}(\pi_1), \wp_{\downarrow}(\pi_2)\rangle"] \arrow[d, mapsto, "\wp_{\downarrow}(\mathsf{u})"'] & (\{S_i\}_i, \{T_i\}_i) \arrow[r, mapsto, "\mathsf{u}"] & \{S_i\}_i \cup \{T_i\}_i \arrow[d, mapsto, "\mu"] \\
\{S_i \cup T_i\}_i \arrow[rr, mapsto, "\mu"] & & (\bigcup_i S_i) \cup (\bigcup_i T_i)
\end{tikzcd}$$

Fig. 7. u is natural, $(\mu \times \mu); \mathsf{u} = \mathsf{u}; \mu$, and $\langle \wp_{\downarrow}(\pi_1), \wp_{\downarrow}(\pi_2)\rangle; \mathsf{u}; \mu = \wp_{\downarrow}(\mathsf{u}); \mu$.

composition). We show the properties:

$$\frac{g : X \to \wp_{\downarrow}(Y) \qquad f : Y \to \wp_{\downarrow}(Y)}{(g; f^*)(x) = \bigcup_{i<\omega} (g; f^i)(x)} \tag{16}$$

$$\frac{f : X \to \wp_{\downarrow}(X) \qquad g : X \to \wp_{\downarrow}(Y)}{(f^*; g)(x) = \bigcup_{i<\omega} (f^i; g)(x)} \tag{17}$$

$$\begin{array}{ll}
(g; f^*)(x) = & (f^*; g)(x) = \\
\bigcup_{y\in g(x)} f^*(y) = & \bigcup_{y\in f^*(x)} g(y) = \\
\bigcup_{y\in g(x)} \bigcup_{i<\omega} f^i(y) = & \bigcup_{y\in \bigcup_{i<\omega} f^i(x)} g(y) = \\
\bigcup_{i<\omega} \bigcup_{y\in g(x)} f^i(y) = & \bigcup_{i<\omega} \bigcup_{y\in f^i(x)} g(y) = \\
\bigcup_{i<\omega} (g; f^i)(x) & \bigcup_{i<\omega} (f^i; g)(x)
\end{array}$$

Now, using Property (16), we have that $(f; f^*)(x) = \bigcup_{i<\omega} (f; f^i)(x) = \bigcup_{i<\omega} f^{i+1}(x) \subseteq f^*(x)$. So, $\eta + f; f^* \le f^*$. Using Property (17), we have that $(f^*; f)(x) = \bigcup_{i<\omega} (f^i; f)(x) = \bigcup_{i<\omega} f^{i+1}(x) \subseteq f^*(x)$. So, $\eta + f^*; f \le f^*$. To show the implication $f; g \le g \Rightarrow f^*; g \le g$, we suppose that $f; g \le g$ and by induction we see that $f^i; g \le g$ for all $i < \omega$. Therefore, using Property (16), we conclude that $(f^*; g)(x) = \bigcup_{i<\omega} (f^i; g)(x) \subseteq g(x)$ and hence $f^*; g \le g$. The implication $g; f \le g \Rightarrow g; f^* \le g$ is shown similarly using Property (17) instead.

It only remains to observe that $\wp_{\downarrow}\mathbb{1} = \{\{\bot_{\mathbb{1}}\}\} \cong \{\bot_{\mathbb{1}}\} = \mathbb{1}$ and the proof is complete. $\square$

B. The Ideal-Completion Monad

An *ω-complete partial order* (ω-CPO) is a partially ordered set $(X, \le)$ that has a least element $\bot_X$ and is *ω-complete* in the sense that every ω-chain (countable chain) $x_0 \le x_1 \le \cdots$ has a supremum $\sup_i x_i$. A function $f : X \to Y$ between ω-CPOs is called *ω-continuous* if it preserves suprema of ω-chains. That is, for every ω-chain $x_0 \le x_1 \le \cdots$ in X, $f(\sup_i x_i) = \sup_i f(x_i)$. An ω-continuous function is monotone. An ω-continuous function is *strict* if $f(\bot) = \bot$. If X, Y are ω-CPOs, then so is their cartesian product $X \times Y$ under the componentwise order:

$$(x_1, x_2) \le (y_1, y_2) \Leftrightarrow x_1 \le y_1 \wedge x_2 \le y_2.$$

The least element is $\bot_{X\times Y} = (\bot_X, \bot_Y)$. For an ω-chain $(x_0, y_0) \le (x_1, y_1) \le \cdots$ in $X \times Y$, $\sup_i (x_i, y_i) =$

$(\sup_i x_i, \sup_i y_i)$. We denote by $[X \to Y]$ the ω-CPO of all ω-continuous functions from X to Y ordered pointwise:

$$f \leq g \Leftrightarrow \forall x \in P.\ f(x) \leq g(x).$$

The bottom element is $\bot_{XY} = \lambda x \in X.\bot_Y$. The supremum of a chain $f_0 \leq f_1 \leq \cdots$ in $[X \to Y]$ is $\sup_i f_i = \lambda x \in X. \sup_i f_i(x)$. For a chain $x_0 \leq x_1 \leq \cdots$ in X, we have that

$$\begin{aligned}(\sup_i f_i)(\sup_j x_j) &= \sup_i (f_i(\sup_j x_j))\\ &= \sup_i \sup_j f_i(x_j)\\ &= \sup_j \sup_i f_i(x_j)\\ &= \sup_j (\sup_i f_i)(x_j),\end{aligned}$$

therefore $\sup_i f_i$ is ω-continuous. The operations ; and $\langle\cdot,\cdot\rangle$ are monotone in all arguments. The ω-continuous functions on ω-CPOs are closed under well-typed composition and pairing and contain all identities and projections. Thus, ω-CPOs and ω-continuous functions form a cartesian category with bottom elements, which we denote by $\mathbf{CPO}$.

Let $(X, \leq)$ be an ω-CPO. A subset $I \subseteq X$ is called an *ideal* of X if it is a non-empty lower set and is closed under suprema of ω-chains. The set of all ideals of X is denoted $\mathfrak{I}X$. We denote by $\mathrm{cl}_X(S)$ the smallest ideal containing $S \subseteq X$. This is an operation of type $\mathrm{cl}_X : \wp X \to \mathfrak{I}X$. We also write $\mathrm{cl}(S)$ instead of $\mathrm{cl}_X(S)$ when no confusion arises. We say that $\mathrm{cl}(S)$ is the *ideal generated by* S. The set $\downarrow x$ is an ideal and we call it the *principal ideal* generated by x. If $x_1 \leq x_2 \leq \ldots$ is an ω-chain in X, then $\mathrm{cl}[\{x_1, x_2, \ldots\}] = \downarrow \sup_i x_i$. The set $\mathfrak{I}X$ of all ideals of an ω-CPO X is a complete lattice w.r.t. set-theoretic inclusion. The meet $\bigwedge$ is set-theoretic intersection and the join $\bigvee$ is the ideal generated by the set-theoretic union. The bottom element is $\{\bot_X\}$ and the top element is X. If I, J are ideals of X, then so is $I \cup J$. In particular, $\mathrm{cl}(S_1 \cup S_2) = \mathrm{cl}(S_1) \cup \mathrm{cl}(S_2)$. Let X, Y be ω-CPOs. If I is an ideal of X and J is an ideal of Y, then $I \times J$ is an ideal of the ω-CPO $X \times Y$. If K is an ideal of $X \times Y$, then the left and right projections of K are ideals of X and Y respectively.

Lemma 21: Let X, Y be ω-CPOs. Then, the following properties hold:

(1) Let $\mathcal{S}$ be a non-empty collection of non-empty subsets of X. Then, $\mathrm{cl}[\bigcup_{S\in\mathcal{S}} \mathrm{cl}(S)] = \mathrm{cl}[\bigcup \mathcal{S}]$.
(2) Let S be a non-empty subset of X that is bounded above by $u \in X$. Then, $\mathrm{cl}(S)$ is also bounded above by u.
(3) Let $\mathcal{S}$ be an ideal of $(\mathfrak{I}(X), \subseteq)$, i.e, $\mathcal{S}$ is in $\mathfrak{I}^2(X)$. Then, $\bigcup \mathcal{S}$ is an ideal of $(X, \leq)$.
(4) Let $\mathcal{S}$ be a non-empty collection of ideals of X. Then, $\bigcup \mathrm{cl}_{\mathfrak{I}X}[\mathcal{S}] = \mathrm{cl}_X[\bigcup \mathcal{S}]$.
(5) If $f : X \to Y$ is an ω-continuous function, then $\mathrm{cl}[f(\mathrm{cl}(S))] = \mathrm{cl}[f(S)]$.
(6) $\mathrm{cl}(A \times B) = \mathrm{cl}(A) \times \mathrm{cl}(B)$.

Proof. Omitted. □

Definition 22 (the ideals endofunctor): We extend $\mathfrak{I}$ to an endofunctor $\mathbf{CPO} \to \mathbf{CPO}$. For $f : X \to Y$ in $\mathbf{CPO}$ define $\mathfrak{I}f : \mathfrak{I}X \to \mathfrak{I}Y$ by

$$(\mathfrak{I}f)(S) := \mathrm{cl}_Y[f(S)] = \mathrm{cl}_Y\{f(x) \mid x \in S\},$$

for $S \in \mathfrak{I}X$. We need to verify that $\mathfrak{I}$ is a functor. Indeed, $\mathfrak{I}(\mathrm{id}_X)(S) = \mathrm{cl}_X(\mathrm{id}_X(S)) = \mathrm{cl}_X S = S$ because S is an ideal, and $(\mathfrak{I}f;\mathfrak{I}g)(S) = \mathrm{cl}(g(\mathrm{cl}(f(S)))) = \mathrm{cl}(g(f(S))) = (\mathfrak{I}(f;g))(S)$ using Lemma 21(5).

Definition 23 (the ideal-completion monad): We define the family of functions $\eta_X : X \to \mathfrak{I}X$ by

$$\eta_X(x) := \downarrow x.$$

For a morphism $f : X \to Y$, we have by monotonicity of f that $f(\downarrow x) = \downarrow f(x)$ is an ideal of $\mathfrak{I}Y$. So, $(\eta_X;\mathfrak{I}f)(x) = (\mathfrak{I}f)(\downarrow x) = \mathrm{cl}[f(\downarrow x)] = \mathrm{cl}[\downarrow f(x)] = \downarrow f(x) = (f;\eta_Y)(x)$. Thus, η is a natural transformation. We define the family of functions $\mu_X : \mathfrak{I}^2X \to \mathfrak{I}X$ by

$$\mu_X(\mathcal{S}) := \bigvee \mathcal{S} = \mathrm{cl}[\bigcup \mathcal{S}] = \bigcup \mathcal{S},$$

by Lemma 21(3), because $\mathcal{S}$ is an ideal of $\mathfrak{I}X$. We argue that μ is a natural transformation: For a function $f : X \to Y$ and $\mathcal{S} \in \mathfrak{I}^2X$, we have that:

$$\begin{aligned}(\mathfrak{I}^2 f;\mu_Y)(\mathcal{S}) &= \bigcup(\mathfrak{I}\mathfrak{I}f)(\mathcal{S}) && [\mu\ \text{def}]\\ &= \bigcup \mathrm{cl}_{\mathfrak{I}Y}[\{(\mathfrak{I}f)(S) \mid S \in \mathcal{S}\}] && [\mathfrak{I}\ \text{def}]\\ &= \mathrm{cl}_Y \bigcup\{(\mathfrak{I}f)(S) \mid S \in \mathcal{S}\} && [\text{L. 21(4)}]\\ &= \mathrm{cl}_Y \textstyle\bigcup_{S\in\mathcal{S}}(\mathfrak{I}f)(S) &&\\ &= \mathrm{cl}_Y \textstyle\bigcup_{S\in\mathcal{S}} \mathrm{cl}_Y[f(S)] && [\mathfrak{I}\ \text{def}]\\ &= \mathrm{cl}_Y \textstyle\bigcup_{S\in\mathcal{S}} f(S) && [\text{L. 21(1)}]\\ &= \mathrm{cl}_Y[f(\bigcup \mathcal{S})] &&\\ (\mu_X;\mathfrak{I}f)(\mathcal{S}) &= (\mathfrak{I}f)(\bigcup \mathcal{S}) && [\mu\ \text{def}]\\ &= \mathrm{cl}_Y[f(\bigcup \mathcal{S})] && [\mathfrak{I}\ \text{def}]\end{aligned}$$

Now, we claim that $(\mathfrak{I}, \eta, \mu)$ is a monad, which we call the *ideal-completion monad* of $\mathbf{CPO}$. It remains to verify the equations $\mathfrak{I}\mu_X;\mu_X = \mu_{\mathfrak{I}X};\mu$, $\eta_{\mathfrak{I}X};\mu_X = \mathrm{id}_{\mathfrak{I}X}$, and $\mathfrak{I}\eta_X;\mu_X = \mathrm{id}_{\mathfrak{I}X}$. These are not hard to prove, given that we have already shown that μ is arbitrary union. We give Kleisli composition in simple set-theoretic terms:

$$\begin{aligned}(f\boldsymbol{;}g)(x) &= (f;\mathfrak{I}g;\mu_Z)(x) && [\boldsymbol{;}\ \text{def}]\\ &= \bigcup(\mathfrak{I}g)(f(x)) && [\mu\ \text{def}]\\ &= \bigcup \mathrm{cl}_{\mathfrak{I}Z}\{g(y) \mid y \in f(x)\} && [\mathfrak{I}\ \text{def}]\\ &= \mathrm{cl}_Z \bigcup\{g(y) \mid y \in f(x)\} && [\text{L. 21(4)}]\\ &= \textstyle\bigvee_{y\in f(x)} g(y).\end{aligned}$$

for $f : X \to \mathfrak{I}Y$ and $g : Y \to \mathfrak{I}Z$.

Theorem 24: Define the family of morphisms $\psi_{X,Y}$ as

$$\begin{aligned}\psi_{X,Y} : \mathfrak{I}X \times \mathfrak{I}Y &\to \mathfrak{I}(X \times Y)\\ (I, J) &\mapsto I \times J\end{aligned}$$

and the family of morphisms u_X as

$$\begin{aligned}\mathsf{u}_X : \mathfrak{I}X \times \mathfrak{I}X &\to \mathfrak{I}X\\ (I, J) &\mapsto I \cup J\end{aligned}$$

Also define the iteration operation by

$$f^* := \textstyle\bigvee_{i<\omega} f^i = \lambda x \in X.\ \bigvee_{i<\omega} f^i(x)$$

Then, the ideal-completion monad $(\mathfrak{I}, \eta, \mu)$ over the category **CPO**, together with the operations $\psi, \mathsf{u}, {}^*$, is a nondeterministic strong monad with iteration.

Proof. Arguing as in the proof of Theorem 20 for the lowerset monad, we see that $(\mathfrak{I}, \eta, \mu), \psi$ is a commutative strong monad with lazy pairs, and $(\mathfrak{I}, \eta, \mu), \mathsf{u}, \perp$ is a nondeterministic monad. For the tensorial strength t induced by ψ, the axiom $\langle \mathrm{id}, f+g\rangle ; t = \langle \mathrm{id}, f\rangle ; t + \langle \mathrm{id}, g\rangle ; t$ holds, as well as the axiom $\mathrm{id} \leq \langle \mathfrak{I}\pi_1, \mathfrak{I}\pi_2\rangle ; \psi$. For the iteration operation, we first show the properties:

$$\frac{g : X \to \mathfrak{I}Y \qquad f : Y \to \mathfrak{I}Y}{(g; f^*)(x) = \bigvee_{i<\omega}(g; f^i)(x)}$$

$$\frac{f : X \to \mathfrak{I}X \qquad g : X \to \mathfrak{I}Y}{(f^*; g)(x) = \bigvee_{i<\omega}(f^i; g)(x)}$$

$$\begin{array}{ll}
(g; f^*)(x) = & [;\ \&\ {}^*\text{ defs}] \\
\bigvee_{y\in g(x)} \bigvee_{i<\omega} f^i(y) = & [\vee\text{ def}] \\
\mathrm{cl}\left[\bigcup_{y\in g(x)} \mathrm{cl}\left(\bigcup_{i<\omega} f^i(y)\right)\right] = & [\text{L. 21(1)}] \\
\mathrm{cl}\left[\bigcup_{y\in g(x)} \bigcup_{i<\omega} f^i(y)\right] = & \\
\mathrm{cl}\left[\bigcup_{i<\omega} \bigcup_{y\in g(x)} f^i(y)\right] = & [\text{L. 21(1)}] \\
\mathrm{cl}\left[\bigcup_{i<\omega} \mathrm{cl}\left(\bigcup_{y\in g(x)} f^i(y)\right)\right] = & [\vee\text{ def}] \\
\mathrm{cl}\left[\bigcup_{i<\omega} \bigvee_{y\in g(x)} f^i(y)\right] = & [;\ \text{def}] \\
\mathrm{cl}\left[\bigcup_{i<\omega} (g; f^i)(x)\right] = & [\vee\text{ def}] \\
\bigvee_{i<\omega} (g; f^i)(x) &
\end{array}$$

$$\begin{array}{ll}
(f^*; g)(x) = & [;, \vee\text{ defs}] \\
\mathrm{cl} \bigcup \{g(y) \mid y \in f^*(x)\} = & [\text{L. 21(4)}] \\
\bigcup \mathrm{cl}\left[\{g(y) \mid y \in f^*(x)\}\right] = & [{}^*, \vee\text{ defs}] \\
\bigcup \mathrm{cl}\left[\{g(y) \mid y \in \mathrm{cl}\left(\bigcup_{i<\omega} f^i(x)\right)\}\right] = & [\text{L. 21(5)}] \\
\bigcup \mathrm{cl}\left[\{g(y) \mid y \in \bigcup_{i<\omega} f^i(x)\}\right] = & [\text{L. 21(4)}] \\
\mathrm{cl}\left[\bigcup \{g(y) \mid y \in \bigcup_{i<\omega} f^i(x)\}\right] = & \\
\mathrm{cl}\left[\bigcup_{i<\omega} \bigcup_{y\in f^i(x)} g(y)\right] = & [\text{L. 21(1)}] \\
\mathrm{cl}\left[\bigcup_{i<\omega} \mathrm{cl}\left[\bigcup_{y\in f^i(x)} g(y)\right]\right] = & [\vee\text{ def}] \\
\mathrm{cl}\left[\bigcup_{i<\omega} \bigvee_{y\in f^i(x)} g(y)\right] = & [;\ \text{def}] \\
\mathrm{cl}\left[\bigcup_{i<\omega} (f^i; g)(x)\right] = & [\vee\text{ def}] \\
\bigvee_{i<\omega} (f^i; g)(x) &
\end{array}$$

Now, the iteration axioms can be shown exactly as in the case of the lowerset monad. □

VI. Typed Kleene Algebra with Products

Let Ω be a set of *atomic types*. Let $\mathbb{1} \notin \Omega$ be a special constant called the *unit type*. The set of *types over* Ω, denoted $\mathsf{Types}(\Omega)$, is the set of terms freely generated by Ω and $\mathbb{1}$ under the binary product type constructor $\times$. The terms of the language are typed. The types of terms are expressions of the form $X \rightharpoonup Y$, where $X, Y \in \mathsf{Types}(\Omega)$. We indicate the type of a term by writing $f : X \rightharpoonup Y$. Some of the terms of the language will be designated as *deterministic*. We indicate the deterministic terms by writing $f : X \twoheadrightarrow Y$. We think of $X \twoheadrightarrow Y$ as a subtype of $X \rightharpoonup Y$ and hence we include the typing rule

$$\frac{f : X \twoheadrightarrow Y}{f : X \rightharpoonup Y}. \tag{18}$$

Let H be a set of *atomic arrows*, each endowed with a fixed type $h : X \rightharpoonup Y$. We write $h : X \twoheadrightarrow Y$ for a deterministic atomic arrow of H. Let

$$\pi_1 : X \times Y \twoheadrightarrow X \qquad \pi_2 : X \times Y \twoheadrightarrow Y \tag{19}$$

$$\mathrm{id} : X \twoheadrightarrow X \qquad \perp : X \twoheadrightarrow Y \tag{20}$$

be special deterministic polymorphic constants called *left projections*, *right projections*, *identities*, and *bottoms*, respectively. Where necessary, we use subscripts or superscripts to denote the specialization at a particular type; e.g., $\mathrm{id}_X : X \twoheadrightarrow X$, $\pi_1^{XY} : X \times Y \twoheadrightarrow X$, or $\perp_{XY} : X \twoheadrightarrow Y$. Let ; and $\langle \cdot, \cdot \rangle$ be polymorphic constructors called *composition* and *pairing*, respectively, satisfying the typing rules

$$\frac{f : X \rightharpoonup Y \qquad g : Y \rightharpoonup Z}{f; g : X \rightharpoonup Z} \tag{21}$$

$$\frac{f : X \twoheadrightarrow Y \qquad g : Y \twoheadrightarrow Z}{f; g : X \twoheadrightarrow Z} \tag{22}$$

$$\frac{f : X \rightharpoonup Y \qquad g : X \rightharpoonup Z}{\langle f, g\rangle : X \rightharpoonup Y \times Z} \tag{23}$$

$$\frac{f : X \twoheadrightarrow Y \qquad g : X \twoheadrightarrow Z}{\langle f, g\rangle : X \twoheadrightarrow Y \times Z} \tag{24}$$

Note that compositions $f; g$ are written in diagrammatic order. Composition and pairing preserve determinism. We add to the language the polymorphic constructors $+$ and *, called *(nondeterministic) choice* and *iteration* respectively, with typing rules

$$\frac{f : X \rightharpoonup Y \qquad g : X \rightharpoonup Y}{f + g : X \rightharpoonup Y} \qquad \frac{f : X \rightharpoonup X}{f^* : X \rightharpoonup X} \tag{25}$$

Choice and iteration introduce nondeterminism.

Definition 25: A *typed Kleene algebra with products* is a multi-sorted algebraic structure

$$\mathcal{K} = \left(\mathcal{K}_0, \times, \mathbb{1}, \mathcal{K}_1, \mathcal{K}_1^d, \mathsf{dom}, \mathsf{cod}, +, ;, {}^*, \perp, \mathrm{id}, \langle \cdot, \cdot \rangle, \pi_1, \pi_2\right),$$

where $\mathcal{K}_0$ is the set of *objects* of $\mathcal{K}$, $\mathcal{K}_1$ is the set of *arrows* or *elements* of $\mathcal{K}$, and $\mathcal{K}_1^d \subseteq \mathcal{K}_1$ is the set of *deterministic arrows/elements*.

- The operations dom and cod are functions (called *domain* and *codomain*) that map arrows to objects.
- The *type* of an element f of $\mathcal{K}$ is the expression $X \rightharpoonup Y$, where $X = \mathsf{dom} f$ and $Y = \mathsf{cod} f$. We write $f : X \rightharpoonup Y$ to denote this. If $f \in \mathcal{K}_1^d$, we write $f : X \twoheadrightarrow Y$.
- The distinguished *product* operation $\times$ is a function $\times : \mathcal{K}_0 \times \mathcal{K}_0 \to \mathcal{K}_0$. The object $\mathbb{1} \in \mathcal{K}_0$ is the distinguished *terminal object* of the structure.

$f+(g+h)=(f+g)+h$	$f;\bot=\bot$
$f+g=g+f$	$\mathrm{id}+f;f^*\le f^*$
$f+\bot=f$	$\mathrm{id}+f^*;f\le f^*$
$f+f=f$	$f;g\le g\to f^*;g\le g$
$f;(g_1+g_2)=f;g_1+f;g_2$	$g;f\le g\to g;f^*\le g$
$(f_1+f_2);g=f_1;g+f_2;g$	
$\langle f,g\rangle;\pi_1=f$	$f=\bot:X\rightharpoonup \mathbb{1}$
$\langle f,g\rangle;\pi_2=g$	$h\le\langle h;\pi_1,h;\pi_2\rangle$
det. h**:** $\langle h;\pi_1,h;\pi_2\rangle=h$	**det.** f**:** $f;\langle g_1,g_2\rangle=\langle f;g_1,f;g_2\rangle$
$(f_1\times f_2);(g_1\times g_2)=$	$\langle f_1+f_2,g\rangle=\langle f_1,g\rangle+\langle f_2,g\rangle$
$(f_1;g_1)\times(f_2;g_2)$	$\langle f,g_1+g_2\rangle=\langle f,g_1\rangle+\langle f,g_2\rangle$

TABLE II
AXIOMS FOR TYPED KLEENE ALGEBRAS WITH PRODUCTS.

- The polymorphic operations and constants $+$, $;$, *, $\bot$, id, $\langle\cdot,\cdot\rangle$, π_1, and π_2 satisfy the expected typing rules.
- Additionally, the structure is a model of the well-typed instances of the axioms in Table II. The partial order $\le$ is induced by $+$: $f\le g$ iff $f+g=g$. The product $\times$ is defined by: $f\times g=\langle\pi_1;f,\pi_2;g\rangle$.

The first group of axioms in Table II are the axioms of Kleene algebra [16, 17] except for the *strictness* axiom $\bot;f=\bot$.

We denote by KA the quasi-equational system of typed Kleene algebra with products. Notice that (up to a slight change in notation to make it less cumbersome), the typing rules and axioms satisfied by a typed Kleene algebra with products are exactly those given in Table I. This means that any small subcategory of the Kleisli category of a nondeterministic strong monad with iteration that is closed under the appropriate operations is a typed Kleene algebra with products.

VII. ITERATION THEORIES

The language of iteration theories consists of *atomic typed actions* $h:n\to m$, where n,m are natural numbers, and polymorphic operation symbols ; (composition), id (identity), ι (injection), $[-,-,\cdots,-]$ (cotupling), $\bot$ (bottom), † (dagger). The typing rules for the language are the following:

$$\mathrm{id}_n:n\to n \qquad \iota_i^n:1\to n \qquad \bot_{nm}:n\to m$$

$$\frac{f:n\to m \qquad g:m\to p}{f;g:n\to p}$$

$$\frac{f_i:1\to m \qquad i=1,\ldots,n}{[f_1,f_2,\ldots,f_n]:n\to m} \qquad \frac{f:n\to n+p}{f^\dagger:n\to p}$$

The *typed terms* $f:n\to m$ of the language are build from the atomic actions and the operation symbols according to the typing rules.

Consider now the category $\mathbf{CPO}$ of ω-CPOs and ω-continuous maps, which will provide the *standard interpretation* for the language of iteration theories. An interpretation $[\![\cdot]\!]$ in $\mathbf{CPO}$ consists of an ω-CPO A and a mapping of every atomic symbol $h:n\to m$ to a morphism $[\![h]\!]:A^m\to A^n$ in $\mathbf{CPO}$, where A^k denotes the k-fold associative cartesian product of A. The identity symbol $\mathrm{id}_n:n\to n$ is interpreted as the identity function $[\![\mathrm{id}_n]\!]:A^n\to A^n$. The injection symbol $\iota_i^n:1\to n$ is interpreted as the i-th projection $[\![\iota_i^n]\!]:A^n\to A$. The bottom symbol $\bot_{nm}:n\to m$ is interpreted as the least function $[\![\bot_{nm}]\!]:A^m\to A^n$ of $\mathbf{CPO}(A^m,A^n)$. Now, ; and $[-,-,\cdots,-]$ are interpreted as function composition and tupling respectively.

$$\frac{f_i:1\to m \qquad i=1,\ldots,n}{[\![[f_1,\ldots,f_n]]\!]:=\lambda\bar{x}\in A^m.\langle[\![f_1]\!](\bar{x}),\ldots,[\![f_n]\!](\bar{x})\rangle}.$$

Every ω-continuous function $\phi:X\to X$ in $\mathbf{CPO}$ has a least fixpoint, which is the supremum of the ω-chain $\bot\le\phi(\bot)\le\phi^2(\bot)\le\cdots\le\phi^n(\bot)\le\cdots$, where $\phi^0=\mathrm{id}$ and $\phi^{n+1}=\phi^n;\phi$. We denote the least fixpoint of ϕ by $\mu(\phi)=\sup_i\phi^i(\bot)$. For an ω-continuous function $\phi:X\times Y\to X$, we define the function $\phi^\dagger:Y\to X$ by $\phi^\dagger(y):=\mu(\phi_y)=\sup_i\phi_y^i(\bot)$, where $\phi_y=\lambda x\in X.\phi(x,y):X\to X$. The function $\phi^\dagger:Y\to X$ is also ω-continuous. We call † the *parametric fixpoint operation*. The operation † is monotone. So, we interpret the dagger symbol † of the language as parametric fixpoint †:

$$\frac{f:n\to n+p}{[\![f^\dagger]\!]=\Big([\![f]\!]:A^n\times A^p\to A^n\Big)^\dagger:A^p\to A^n}.$$

We have thus defined for every typed term $f:n\to m$ its interpretation $[\![f]\!]:A^m\to A^n$.

Every homset $\mathbf{CPO}(X,Y)$ is equipped with the pointwise partial order $\le$. For terms s,t we write $\mathbf{CPO}\models s\le t$ to mean that $[\![s]\!]\le[\![t]\!]$ for every interpretation $[\![\cdot]\!]$ of the language in $\mathbf{CPO}$. Define $\mathsf{Th}(\mathbf{CPO})$ to be the set of all valid inequalities over $\mathbf{CPO}$, that is,

$$\mathsf{Th}(\mathbf{CPO}):=\{s\le t\mid \mathbf{CPO}\models s\le t\},$$

where s,t are terms of the language of iteration theories. $\mathsf{Th}(\mathbf{CPO})$ is the "(in)equational theory of iteration", in the words of Ésik [4].

Theorem 26 (Ésik [4]): Let Park be the universal Horn system (with equality) that includes the following:

(1) Axioms for categories.

$$\frac{f:n\to m \qquad g:m\to p \qquad h:p\to q}{(f;g);h=f;(g;h):n\to q}$$

$$\frac{f:n\to m}{\mathrm{id};f=f:n\to m} \qquad \frac{f:n\to m}{f;\mathrm{id}=f:n\to m}$$

(2) Axioms asserting that ι, $[-,-,\cdots,-]$, and $\bot$ give associative categorical coproducts.

$$\frac{}{\iota_1^1=\mathrm{id}_1:1\to 1} \qquad \frac{f_k:1\to m \qquad k=1,\ldots,n}{\iota_i^n;[f_1,f_2,\ldots,f_n]=f_i:1\to m}$$

$$\frac{h:n\to p}{[\iota_1^n;h,\iota_2^n;h,\ldots,\iota_n^n;h]=h:n\to p}$$

(3) Axioms stating that $\le$ is a partial order.
(4) Axioms stating that ; and $[-,-,\cdots,-]$ are motonone in all arguments w.r.t. $\le$.
(5) Axiom stating that $\bot_{nm}:n\to m$ is the least element of $\mathrm{Hom}(n,m)$.

$$\frac{f:X\to Y}{\bot_{XY}\le f:X\to Y}$$

(6) Axioms for the dagger operation:

$$\frac{f : n \to n+p}{f; [f^\dagger, \mathrm{id}_p] \leq f^\dagger : n \to p}$$

$$\frac{f : n \to n+p \qquad g : n \to p}{f; [g, \mathrm{id}_p] \leq g \Rightarrow f^\dagger \leq g}$$

$$\frac{f : n \to n+p \qquad g : p \to q}{f^\dagger; g \leq [f; (\mathrm{id}_n \oplus g)]^\dagger : n \to q}$$

where the copairing operation $[-,-]$ is induced by the cotupling operation $[-,-,\ldots,-]$ in the obvious way. The first two axioms are called *pre-fixpoint inequality* and *least pre-fixpoint implication* or *Park induction rule* respectively. The last one is called *parameter inequality*.

Park axiomatizes $\mathsf{Th}(\mathbf{CPO})$, that is, $\mathbf{CPO} \models t_1 \leq t_2$ iff $\mathsf{Park} \vdash t_1 \leq t_2$, for all terms t_1, t_2 in the language of iteration theories.

A. The Opposite Category

The choice of a language with coproducts and copairing/injection symbols is confusing, because the standard models we are interested in here are models of functions where the symbols are interpreted as products and pairing/projections respectively. Moreover, there is no reason to collapse isomorphic products $X \times (Y \times Z) \cong (X \times Y) \times Z$. In fact, this would only complicate the technical presentation of our proofs.

So, we consider for the rest of the paper that the language of iteration theories is instead as follows: For a set Ω of atomic types, let $\mathsf{Types}(\Omega)$ be the set freely generated by Ω, $\mathbb{1} \notin \Omega$, and the product constructor $\times$. The terms of the language are typed, e.g., $f : X \to Y$, where $X, Y \in \mathsf{Types}(\Omega)$. Each atomic arrow has a fixed type $h : X \to Y$. We have polymorphic constants π_1^{XY}, π_2^{XY}, id_X, $\bot_{XY}$ and polymorphic constructors $;$, $\langle\cdot,\cdot\rangle$, $\dagger$. The typing rules are as usual with the exception of the rule for $\dagger$:

$$\frac{f : X \times Y \to X}{f^\dagger : Y \to X}.$$

Now, a standard interpretation $[\![\cdot]\!]$ in $\mathbf{CPO}$ assigns an ω-CPO to each base type and an ω-continuous function to each atomic action. This extends in the obvious way to all terms of the language. In particular, the dagger symbol is interpreted as parametric fixpoint, e.g., $[\![f^\dagger]\!] = [\![f]\!]^\dagger : [\![Y]\!] \to [\![X]\!]$ for $f : X \times Y \to X$. The completeness theorem (Theorem 26) can now be appropriately restated. The axioms for the dagger, for example, become:

$$\frac{f : X \times Y \to X}{\langle f^\dagger, \mathrm{id}_Y\rangle; f \leq f^\dagger : Y \to X}$$

$$\frac{f : X \times Y \to X \qquad g : Y \to X}{\langle g, \mathrm{id}_Y\rangle; f \leq g \Rightarrow f^\dagger \leq g}$$

$$\frac{f : X \times Y \to X \qquad g : Z \to Y}{g; f^\dagger \leq [(\mathrm{id}_X \times g); f]^\dagger : Z \to X : Z \to X}$$

VIII. Embedding the Equational Theory of Iteration in KA

We augment the system of KA that we presented in Section VI with an additional typing rule about the preservation of determinism:

$$\frac{g : X \rightarrowtriangle Y \qquad f : Y \rightarrowtriangle Y \qquad g \leq g; f : X \rightharpoonup Y}{g; f^* : X \rightarrowtriangle Y}. \quad (26)$$

We note that this rule is not valid in all Kleene algebras, but it is valid in the Kleisli category $\mathbf{CPO}_{\mathfrak{I}}$ of the ideal-completion monad over $\mathbf{CPO}$. For a term $f : X \times Y \rightharpoonup X$ of KA, we define the abbreviation

$$f^\ddagger := \langle \bot_{YX}, \mathrm{id}_Y\rangle; \langle f, \pi_2\rangle^*; \pi_1 : Y \rightharpoonup X.$$

We call ‡ the *derived dagger operation* in KA. Using the typing rule (26) we can derive in KA the rule

$$\frac{f : X \times Y \rightarrowtriangle X}{f^\ddagger : Y \rightarrowtriangle X} \quad (27)$$

as follows: Since $\bot, \mathrm{id}$ are deterministic, then so is $\langle \bot, \mathrm{id}\rangle$. Similarly, $\langle f, \pi_2\rangle$ is deterministic. Now, from $\langle \bot, \mathrm{id}\rangle \leq \langle \bot, \mathrm{id}\rangle; \langle f, \pi_2\rangle = \langle\langle \bot, \mathrm{id}, \rangle; f, \mathrm{id}\rangle$ and rule (26) we conclude that $\langle \bot, \mathrm{id}\rangle; \langle f, \pi_2\rangle^*$ and hence $f^\ddagger$ is deterministic. Rule (27) states that the derived dagger operation preserves determinism. In fact, for our purposes (as will become apparent in the proofs later) we can augment the system of KA with the weaker rule (27) instead of the rule (26).

We define a translation $[\cdot]$ from the language of iteration theories to the language of Kleene algebra with products. All atomic action symbols and atomic constants are tranlated as deterministic symbols with the same type. E.g., for $h : X \to Y$ we have $[h] = h : X \rightarrowtriangle Y$, and $[\pi_1] = \pi_1 : X \times Y \rightarrowtriangle X$. The dagger is translated as

$$[f^\dagger] := f^\ddagger = \langle \bot, \mathrm{id}\rangle; \langle [f], \pi_2\rangle^*; \pi_1 : Y \rightarrowtriangle X.$$

The translation function $[\cdot]$ commutes with the rest of the operation symbols.

Theorem 27 (Completeness): Let $t \leq t'$ be an inequality in the language of Park. Then, $\mathbf{CPO} \models t \leq t'$ implies that $\mathsf{KA} \vdash [t] \leq [t']$.

Proof. We show that the rule

$$\frac{f : X \times Y \rightarrowtriangle X}{\langle f^\ddagger, \mathrm{id}_Y\rangle; f \leq f^\ddagger : Y \rightarrowtriangle X} \quad (28)$$

is provable in KA. We claim that $\langle f, \pi_2\rangle^*; \pi_2 = \pi_2$. From $\mathrm{id}_{X \times Y} \leq \langle f, \pi_2\rangle^*$ and monotonicity of ; we have that $\pi_2 \leq \langle f, \pi_2\rangle^*; \pi_2$. For the other direction, notice that

$$\begin{aligned} \langle f, \pi_2\rangle; \pi_2 = \pi_2 &\implies \langle f, \pi_2\rangle; \pi_2 \leq \pi_2 \\ &\implies \langle f, \pi_2\rangle^*; \pi_2 \leq \pi_2. \end{aligned}$$

So, $\langle f, \pi_2\rangle^*; \pi_2 = \pi_2$. From the equation

$$\langle \bot, \mathrm{id}\rangle; \langle f, \pi_2\rangle^*; \pi_2 = \langle \bot, \mathrm{id}\rangle; \pi_2 = \mathrm{id},$$

the equation $\langle \bot, \mathrm{id}\rangle ; \langle f, \pi_2\rangle^* ; \pi_1 = f^\ddagger$, and using the uniqueness axiom we obtain that $\langle f^\ddagger, \mathrm{id}\rangle = \langle \bot, \mathrm{id}\rangle ; \langle f, \pi_2\rangle^*$. Now, we have that

$$\begin{aligned}
&\langle f, \pi_2\rangle^* ; \langle f, \pi_2\rangle \leq \langle f, \pi_2\rangle^* \implies && \text{[mon.]}\\
&\langle \bot, \mathrm{id}\rangle ; \langle f, \pi_2\rangle^* ; \langle f, \pi_2\rangle \leq \langle \bot, \mathrm{id}\rangle ; \langle f, \pi_2\rangle^* \implies &&\\
&\langle f^\ddagger, \mathrm{id}\rangle ; \langle f, \pi_2\rangle \leq \langle f^\ddagger, \mathrm{id}\rangle \implies && [-;\pi_1]\\
&\langle f^\ddagger, \mathrm{id}\rangle ; \langle f, \pi_2\rangle ; \pi_1 \leq \langle f^\ddagger, \mathrm{id}\rangle ; \pi_1 \implies &&\\
&\langle f^\ddagger, \mathrm{id}\rangle ; f \leq f^\ddagger, &&
\end{aligned}$$

which establishes rule (28). Now, we show that the rule

$$\frac{f : X \times Y \rightharpoonup X \qquad g : Y \rightharpoonup X}{\langle g, \mathrm{id}_Y\rangle ; f \leq g \Rightarrow f^\ddagger \leq g} \tag{29}$$

is provable in KA. Suppose that $\langle g, \mathrm{id}_Y\rangle ; f \leq g$. From our assumption, the inequality $\mathrm{id}_Y \leq \mathrm{id}_Y$, and from monotonicity of tupling we have that

$$\begin{aligned}
&\langle \langle g, \mathrm{id}\rangle ; f, \mathrm{id}\rangle \leq \langle g, \mathrm{id}\rangle \implies\\
&\langle \langle g, \mathrm{id}\rangle ; f, \langle g, \mathrm{id}\rangle ; \pi_2\rangle \leq \langle g, \mathrm{id}\rangle \implies\\
&\langle g, \mathrm{id}\rangle ; \langle f, \pi_2\rangle \leq \langle g, \mathrm{id}\rangle \implies\\
&\langle g, \mathrm{id}\rangle ; \langle f, \pi_2\rangle^* \leq \langle g, \mathrm{id}\rangle .
\end{aligned}$$

Now, $\bot_{YX} \leq g$ implies that $\langle \bot_{YX}, \mathrm{id}_Y\rangle \leq \langle g, \mathrm{id}_Y\rangle$. So,

$$\begin{aligned}
f^\ddagger &= \langle \bot_{YX}, \mathrm{id}_Y\rangle ; \langle f, \pi_2\rangle^* ; \pi_1\\
&\leq \langle g, \mathrm{id}_Y\rangle ; \langle f, \pi_2\rangle^* ; \pi_1\\
&\leq \langle g, \mathrm{id}_Y\rangle ; \pi_1 = g,
\end{aligned}$$

which establishes rule (29). We show that the rule

$$\frac{f : X \times Y \to X \qquad g : Z \to Y}{g; f^\ddagger \leq [(\mathrm{id}_X \times g); f]^\ddagger : Z \to X} \tag{30}$$

is provable in KA. First, we notice that

$$\begin{aligned}
g; \langle \bot_{YX}, \mathrm{id}_Y\rangle = \langle g; \bot_{YX}, g; \mathrm{id}_Y\rangle = \langle \bot_{ZX}, g\rangle =\\
\langle \bot_{ZX}; \mathrm{id}_X, \mathrm{id}_Z; g\rangle = \langle \bot_{ZX}, \mathrm{id}_Z\rangle ; (\mathrm{id}_X \times g) .
\end{aligned}$$

It follows that

$$\begin{aligned}
g; f^\ddagger &= g; \langle \bot_{YX}, \mathrm{id}_Y\rangle ; \langle f, \pi_2\rangle^* ; \pi_1\\
&= \langle \bot_{ZX}, \mathrm{id}_Z\rangle ; (\mathrm{id}_X \times g); \langle f, \pi_2\rangle^* ; \pi_1 .
\end{aligned}$$

Now, we have that

$$\begin{aligned}
(\mathrm{id}_X \times g); \langle f, \pi_2\rangle &= \langle (\mathrm{id}_X \times g); f, (\mathrm{id}_X \times g); \pi_2\rangle\\
&= \langle (\mathrm{id}_X \times g); f, \pi_2; g\rangle\\
&= \langle (\mathrm{id}_X \times g); f; \mathrm{id}_X, \pi_2; g\rangle\\
&= \langle (\mathrm{id}_X \times g); f, \pi_2\rangle ; (\mathrm{id}_X \times g).
\end{aligned}$$

Define $\psi = \langle (\mathrm{id}_X \times g); f, \pi_2\rangle^* ; (\mathrm{id}_X \times g)$ and observe that

$$\begin{aligned}
\psi; \langle f, \pi_2\rangle &= \langle (\mathrm{id}_X \times g); f, \pi_2\rangle^* ; (\mathrm{id}_X \times g); \langle f, \pi_2\rangle\\
&= \langle (\mathrm{id}_X \times g); f, \pi_2\rangle^* ; \langle (\mathrm{id}_X \times g); f, \pi_2\rangle ; (\mathrm{id}_X \times g)\\
&\leq \langle (\mathrm{id}_X \times g); f, \pi_2\rangle^* ; (\mathrm{id}_X \times g)\\
&= \psi.
\end{aligned}$$

TABLE III
AXIOMS OF PARK AND KA.

Park	KA
$f \leq g \wedge g \leq h \Rightarrow f \leq h$ $f \leq g \wedge g \leq f \Rightarrow f = g$ $\bot \leq f$ $f \leq f$	$f + (g + h) = (f + g) + h$ $f + g = g + f$ $f + \bot = f$ $f + f = f$
$f; (g; h) = (f; g); h$ $\mathrm{id}; f = f$ $f; \mathrm{id} = f$	$f; (g; h) = (f; g); h$ $\mathrm{id}; f = f$ $f; \mathrm{id} = f$
$g_1 \leq g_2 \Rightarrow f; g_1 \leq f; g_2$ $f_1 \leq f_2 \Rightarrow f_1; g \leq f_2; g$	$f; (g_1 + g_2) = f; g_1 + f; g_2$ $(f_1 + f_2); g = f_1; g + f_2; g$
	$f; \bot = \bot$
$\langle f^\dagger, \mathrm{id}_Y\rangle ; f \leq f^\dagger$ $\langle g, \mathrm{id}_Y\rangle ; f \leq g \Rightarrow f^\dagger \leq g$ $g; f^\dagger \leq [(\mathrm{id}_X \times g); f]^\dagger$	$\mathrm{id} + f; f^* \leq f^*$ $\mathrm{id} + f^*; f \leq f^*$ $f; g \leq g \to f^*; g \leq g$ $g; f \leq g \to g; f^* \leq g$
$f = \bot_X : X \to 1$ $\langle f, g\rangle ; \pi_1 = f$ $\langle f, g\rangle ; \pi_2 = g$ $\langle h; \pi_1, h; \pi_2\rangle = h$	$f = \bot_X : X \to 1$ $\langle f, g\rangle ; \pi_1 = f$ $\langle f, g\rangle ; \pi_2 = g$ $\langle h; \pi_1, h; \pi_2\rangle = h$ (det. h)
	$(f_1 \times f_2); (g_1 \times g_2) =$ $(f_1; g_1) \times (f_2; g_2)$ $h \leq \langle h; \pi_1, h; \pi_2\rangle$ $f; \langle g_1, g_2\rangle = \langle f; g_1, f; g_2\rangle$ (det. f)
$f_1 \leq f_2 \Rightarrow \langle f_1, g\rangle \leq \langle f_2, g\rangle$ $g_1 \leq g_2 \Rightarrow \langle f, g_1\rangle \leq \langle f, g_2\rangle$	$\langle f_1 + f_2, g\rangle = \langle f_1, g\rangle + \langle f_2, g\rangle$ $\langle f, g_1 + g_2\rangle = \langle f, g_1\rangle + \langle f, g_2\rangle$

Moreover, $\mathrm{id}_{X\times Z} \leq \langle (\mathrm{id}_X \times g); f, \pi_2\rangle^*$ and by monotonicity $(\mathrm{id}_X \times g) \leq \psi$. We have thus shown that

$$(\mathrm{id}_X \times g) + \psi; \langle f, \pi_2\rangle \leq \psi$$

from which we deduce that $(\mathrm{id}_X \times g); \langle f, \pi_2\rangle^* \leq \psi$. Finally, we have that

$$\begin{aligned}
g; f^\ddagger &= \langle \bot_{ZX}, \mathrm{id}_Z\rangle ; (\mathrm{id}_X \times g); \langle f, \pi_2\rangle^* ; \pi_1\\
&\leq \langle \bot_{ZX}, \mathrm{id}_Z\rangle ; \psi; \pi_1\\
&= \langle \bot_{ZX}, \mathrm{id}_Z\rangle ; \langle (\mathrm{id}_X \times g); f, \pi_2\rangle^* ; (\mathrm{id}_X \times g); \pi_1\\
&= \langle \bot_{ZX}, \mathrm{id}_Z\rangle ; \langle (\mathrm{id}_X \times g); f, \pi_2\rangle^* ; \pi_1\\
&= [(\mathrm{id}_X \times g); f]^\ddagger,
\end{aligned}$$

which establishes rule (30).

Suppose that the inequality $t \leq t'$ is valid in $\mathbf{CPO}$. From completeness of Park (Theorem 26) we have that Park $\vdash t \leq t'$. It suffices to show that the (translations of) the axioms of Park can be shown in KA. From rules (28), (29), (30) we get that the axioms for the dagger are all provable in KA. The rest of the axioms of Park can be easily proved from corresponding axioms of KA as indicated in Table III. □

We consider now the Kleisli category $\mathbf{CPO}_{\mathfrak{I}}$ of the ideal-completion monad $\mathfrak{I}$ over $\mathbf{CPO}$. We saw in Theorem 24 that $\mathfrak{I}$ is a nondeterministic strong monad with iteration. Therefore, according to Theorem 16, the Kleisli category $\mathbf{CPO}_{\mathfrak{I}}$, with the constants and operations

$$\times \quad \mathbb{1} \quad \eta \quad ; \quad \varpi_1 \quad \varpi_2 \quad \langle\!\langle \cdot, \cdot \rangle\!\rangle \quad \bot \quad + \quad {}^*$$

that we have defined in it, satisfies all the axioms of typed Kleene algebra with products: $\mathbf{CPO}_{\mathfrak{I}} \models \mathsf{KA}$. Moreover, the order induced by nondeterministic choice is the same as the pointwise order of the homsets (which is in turn induced by set-theoretic inclusion on ideals). We have already seen that the unit functor $H = (-;\eta) : \mathbf{CPO} \to \mathbf{CPO}_{\mathfrak{I}}$ commutes with identities, composition, projections, pairing, and bottoms:

$$H\mathrm{id} = \eta \qquad H(f;g) = Hf\,\mathbf{;}\,Hg \qquad H\pi_1 = \varpi_1$$
$$H\pi_2 = \varpi_2 \qquad H\langle f,g\rangle = \langle\!\langle Hf, Hg\rangle\!\rangle \qquad H\bot = \bot$$

Now, we will define a *Kleisli dagger operation* ‡ in $\mathbf{CPO}_{\mathfrak{I}}$

$$\frac{f : X \times Y \to \mathfrak{I}X \text{ in } \mathbf{CPO}_{\mathfrak{I}}}{f^\ddagger : Y \to \mathfrak{I}X \text{ in } \mathbf{CPO}_{\mathfrak{I}}} \qquad f^\ddagger(y) := \bigvee_{i<\omega} f_y^i(\bot)$$

where $f_y^0 = \eta_X$, $f_y^{i+1} = f_y^i\,\mathbf{;}\,f_y$, and $f_y := \lambda x \in X.\ f(x,y) : X \to \mathfrak{I}X$.

Lemma 28: For functions $g : X \to Y$ and $f : Y \to Y$ in $\mathbf{CPO}$ with $g \leq g; f$, we have that

$$Hg\,\mathbf{;}\,(Hf)^* = \lambda x \in X.\ {\downarrow}\mathrm{sup}_i(g; f^i)(x) : X \to \mathfrak{I}Y.$$

In particular, the typing rule (26) is valid in $\mathbf{CPO}_{\mathfrak{I}}$. Moreover, the rule

$$\frac{f : X \times Y \to X \text{ in } \mathbf{CPO}}{H(f^\dagger) = (Hf)^\ddagger : Y \to \mathfrak{I}X}$$

is valid, that is, H commutes with the dagger operation.

Proof. Let $x \in X$. Define $y_i = (g; f^i)(x)$ for all $i < \omega$. An induction argument gives us that $g; f^i \leq g; f^{i+1}$, which implies that $y_i \leq y_{i+1}$. But Y is an ω-CPO, which means that the ω-chain $y_0 \leq y_1 \leq y_2 \leq \cdots$ has a supremum

$$y = \mathrm{sup}_i\, y_i = \mathrm{sup}_i (g; f^i)(x)$$

in Y. Now, notice that

$$\begin{aligned}[Hg\,\mathbf{;}\,(Hf)^*](x) &= \textstyle\bigvee_{i<\omega}[Hg\,\mathbf{;}\,(Hf)^i](x)\\ &= \textstyle\bigvee_{i<\omega}[H(g; f^i)](x)\\ &= \textstyle\bigvee_{i<\omega} {\downarrow}(g; f^i)(x),\end{aligned}$$

which is equal to $\bigvee_{i<\omega} {\downarrow}y_i = {\downarrow}\mathrm{sup}_i\, y_i = {\downarrow}y$. The validity of the typing rule (26) in $\mathbf{CPO}_{\mathfrak{I}}$ is an immediate consequence. For the second part of the lemma, let $y \in Y$. We first observe that

$$(Hf)_y = \lambda x.(Hf)(x,y) = \lambda x.{\downarrow}f(x,y) = f_y; \eta_X : X \to \mathfrak{I}X.$$

Since H commutes with composition, a simple induction argument gives us that $(Hf)_y^i = H(f_y^i)$. By monotonicity of f, we also have that $f_y^i(\bot) \leq f_y^{i+1}(\bot)$ for all $i < \omega$, so $\bot \leq f_y(\bot) \leq f_y^2(\bot) \leq \cdots$ is an ω-chain in X. Now,

$$\begin{aligned}(Hf)^\ddagger(y) = \textstyle\bigvee_{i<\omega}(Hf)_y^i(\bot) = \bigvee_{i<\omega} {\downarrow}f_y^i(\bot) =\\ {\downarrow}\mathrm{sup}_i\, f_y^i(\bot) = {\downarrow}f^\dagger(y) = (Hf^\dagger)(y),\end{aligned}$$

and therefore $Hf^\dagger = (Hf)^\ddagger$. □

We have thus established that H is an *embedding* $\mathbf{CPO} \to \mathbf{CPO}_{\mathfrak{I}}$ that respects all the operations. Additionally, it is order-preserving: $f \leq g$ in $\mathbf{CPO}$ iff $Hf \leq Hg$ in $\mathbf{CPO}_{\mathfrak{I}}$. A consequence of this fact is that the rules

$$\frac{f : X \times Y \rightharpoonup X}{\langle\!\langle f^\ddagger, \eta_Y\rangle\!\rangle\,\mathbf{;}\,f \leq f^\ddagger : Y \rightharpoonup X}$$

$$\frac{f : X \times Y \rightharpoonup X \qquad g : Y \rightharpoonup X}{\langle\!\langle g, \eta_Y\rangle\!\rangle\,\mathbf{;}\,f \leq g \Rightarrow f^\ddagger \leq g}$$

$$\frac{f : X \times Y \rightharpoonup X \qquad g : Z \rightharpoonup Y}{g\,\mathbf{;}\,f^\ddagger \leq [(\eta_X \otimes g)\,\mathbf{;}\,f]^\ddagger : Z \rightharpoonup X}$$

are satisfied in $\mathbf{CPO}_{\mathfrak{I}}$.

Theorem 29 (Soundness): Let $t \leq t'$ be an inequality in the language of Park. Then, $\mathsf{KA} \vdash [t] \leq [t']$ implies that $\mathbf{CPO} \models t \leq t'$.

Proof. First, we show that for an arbitrary deterministic function $\phi = Hf : X \times Y \to \mathfrak{I}X$ it holds that

$$\phi^\ddagger = \langle\!\langle \bot_{YX}, \eta_y\rangle\!\rangle\,\mathbf{;}\,\langle\!\langle \phi, \varpi_2\rangle\!\rangle^*\,\mathbf{;}\,\varpi_1.$$

In other words, the translation of the dagger using star is valid in $\mathbf{CPO}_{\mathfrak{I}}$. Now, we have that $\langle\!\langle \phi, \varpi_2\rangle\!\rangle^i(x,y) = \langle\!\langle Hf, H\pi_2\rangle\!\rangle^i(x,y) = [H\langle f, \pi_2\rangle]^i(x,y) = {\downarrow}\langle f, \pi_2\rangle^i(x,y) = {\downarrow}(f_y^i(x), y)$. So,

$$\begin{aligned}&[\langle\!\langle \bot_{YX}, \eta_y\rangle\!\rangle\,\mathbf{;}\,\langle\!\langle \phi, \varpi_2\rangle\!\rangle^*](y) =\\ &\textstyle\bigvee_{i<\omega}[\langle\!\langle \bot_{YX}, \eta_y\rangle\!\rangle\,\mathbf{;}\,\langle\!\langle \phi, \varpi_2\rangle\!\rangle^i](y) =\\ &\textstyle\bigvee_{i<\omega}[\langle \bot_{YX}, \mathrm{id}_Y\rangle; \langle\!\langle \phi, \varpi_2\rangle\!\rangle^i](y) =\\ &\textstyle\bigvee_{i<\omega}\langle\!\langle \phi, \varpi_2\rangle\!\rangle^i(\bot_X, y) =\\ &\textstyle\bigvee_{i<\omega} {\downarrow}(f_y^i(\bot), y),\end{aligned}$$

which is equal to ${\downarrow}\mathrm{sup}_i(f_y^i(\bot), y)$, because $\bot \leq f_y(\bot) \leq f_y^2(\bot) \leq \cdots$ is an ω-chain. But $\mathrm{sup}_i(f_y^i(\bot), y) = (\mathrm{sup}_i\, f_y^i(\bot), y) = (f^\dagger(y), y)$, so

$$\begin{aligned}&[\langle\!\langle \bot_{YX}, \eta_y\rangle\!\rangle\,\mathbf{;}\,\langle\!\langle \phi, \varpi_2\rangle\!\rangle^*\,\mathbf{;}\,\varpi_1](y) =\\ &\qquad\qquad \varpi_1(f^\dagger(y), y) = {\downarrow}f^\dagger(y) = \phi^\ddagger(y).\end{aligned}$$

Now, we are ready to prove the theorem. Suppose that $\mathsf{KA} \vdash [t] \leq [t']$. Since KA is sound for $\mathbf{CPO}_{\mathfrak{I}}$, we have that $\mathbf{CPO}_{\mathfrak{I}} \models [t] \leq [t']$. But we showed before that the translation of the dagger using star is valid in $\mathbf{CPO}_{\mathfrak{I}}$. Also, the order induced by $+$ is equal to the pointwise order of the homsets. So, we can write $\mathbf{CPO}_{\mathfrak{I}} \models t \leq t'$. But $\mathbf{CPO}$ is embedded in $\mathbf{CPO}_{\mathfrak{I}}$ through the embedding $H = (-;\eta)$. So, $\mathbf{CPO} \models t \leq t'$ and the proof is complete. □

IX. Related Work

The work by Goncharov [15] is closely related to ours. He defines *additive (strong) monads* and *Kleene monads* axiomatically. Calculi for an extended metalanguage of effects are defined and completeness/incompleteness results are obtained. Our notion of a "nondeterministic strong monad with iteration" is different from that of a Kleene monad: we consider non-strict programs that form lazy pairs, and the axioms that we give for iteration are quasi-equational.

The work on Hoare powerdomains [18, 19], which give models of angelic nondeterministic computations, is also related. The (lower) Hoare powerdomain of a domain is formed by taking all the ideals of the domain. An ideal is typically defined to be a nonempty Scott-closed (down-closed and containing suprema of directed subsets) subset of the domain. In the present work, we identify models of the axiomatically defined "nondeterministic monad with iteration," which are similar to and simpler than the construction of Hoare powerdomains over DCPOs. We first identify a simple model: the lowerset monad over the category of posets with bottom elements. Then, we also prove that the ideal-completion monad over the category of ω-CPOs is a model.

There is a long line of work, primarily by Stephen Bloom and Zoltán Ésik, under the name of "iteration theories" or the "(in)equational theory of iteration" [1, 3, 20–27], which is intimately related to the work on Kleene algebra (KA) [9, 16, 17, 28–30] in general and the present work in particular. The axioms of iteration theories capture the equational properties of fixpoints in several classes of structures relevant to computer science. For example, they capture the equational theory of ω-continuous functions between ω-CPOs, where the algebraic signature includes symbols for composition, pairing, and parametric fixpoints. A multitude of different equational axiomatizations have been considered in the literature, all of which require substantial effort to parse and understand. By allowing quasi-equations, simpler axiomatizations can be found. Many examples of iteration theories involve functions on posets, so it is a natural question to look for complete axiomatizations of the valid inequalities over classes of structures that are of interest, e.g., structures of ω-continuous functions over ω-CPOs. One such universal Horn axiomatization is given in [4]. This axiomatization includes two inequalities and one implication for the parametric fixpoint operation, which are both intuitive and easy to memorize. We note that in all this work on iteration theories, the issue of nondeterminism, which is central in the present work, in not handled at all.

Of particular relevance to the relationship between iteration theories and Kleene algebra are the works on the so-called "matrix theories" [1, 3, 31, 32]. They are cartesian theories in which homsets are commutative monoids with respect to an operation $+$, which distributes over composition. This also induces cocartesian structure and allows an easy translation between the dagger (parametric fixpoint) operation and Kleene star. However, this translation is not valid in the classes of structures we consider. In particular, the $+$ symbol cannot be interpreted as nondeterministic choice: for example, the stipulated axioms imply the property $\langle f, \perp\rangle + \langle \perp, g\rangle = \langle f, g\rangle$, which is not meaningful for programs. The translation of the dagger operation in the language of KA that we give is different and is in fact valid in the class of matrix theories as well.

Our work here builds directly upon the existing work on Kleene algebra [9, 16, 17, 28–30]. The crucial axioms for the iteration operation * are taken from [16]. The system of KA we present is a typed Kleene algebra in the sense of [12] extended with products that satisfy weaker axioms than those of categorical products.

X. Conclusion and Future Work

The present work was motivated by the desire to reconcile the notions of iteration captured by the star operation * of KA and the dagger operation $\dagger$ of IT. We presented and investigated a system of typed KA with products, in which the notion of a deterministic program turns out to be of importance. We work in the framework of cartesian categories combined with commutative strong monads to treat (angelic) nondeterminism. We have adapted an axiomatization of commutative strong monads that can be found in the work of Kock in the 70's [13, 14] to our setting. We have also described three equivalent ways, in the presence of cartesian structure, of capturing nondeterminism. We have shown that two concrete monads, the monad of lowersets of pointed posets and the monad of ideals of ω-CPOs, are models. The main technical result of our paper is a translation of $\dagger$ in terms of * that gives an embedding of the equational theory of $\dagger$ in our system.

The present work has been a first step in presenting a higher-order system of typed Kleene algebra. We would like to investigate what properties of recursion can be captured in such a higher-order system and how this would relate to the investigations of [24].

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Appendix

$$\begin{tikzcd} X \times PY \arrow[r, "t_{X,Y}"] \arrow[dr, "\pi_2"'] & P(X \times Y) \arrow[d, "P\pi_2"] \\ & PY \end{tikzcd} \qquad \begin{tikzcd} X \times Y \arrow[r, "\mathrm{id} \times \eta_Y"] \arrow[dr, "\eta_{X \times Y}"'] & X \times PY \arrow[d, "t_{X,Y}"] \\ & P(X \times Y) \end{tikzcd}$$

$$\begin{tikzcd} (X \times Y) \times PZ \arrow[r, "\alpha"] \arrow[d, "t_{X \times Y, Z}"'] & X \times (Y \times PZ) \arrow[r, "\mathrm{id} \times t_{Y,Z}"] & X \times P(Y \times Z) \arrow[d, "t_{X, Y \times Z}"] \\ P((X \times Y) \times Z) \arrow[rr, "P\alpha"] & & P(X \times (Y \times Z)) \end{tikzcd}$$

$$\begin{tikzcd} X \times P^2 Y \arrow[r, "t_{X,PY}"] \arrow[d, "\mathrm{id} \times \mu_Y"'] & P(X \times PY) \arrow[r, "Pt_{X,Y}"] & P^2(X \times Y) \arrow[d, "\mu_{X \times Y}"] \\ X \times PY \arrow[rr, "t_{X,Y}"] & & P(X \times Y) \end{tikzcd}$$

Fig. 8. t is tensorial strength for the monad (P, η, μ).

$$\begin{tikzcd} PX \times Y \arrow[r, "\tau_{X,Y}"] \arrow[dr, "\pi_1"'] & P(X \times Y) \arrow[d, "P\pi_1"] \\ & PX \end{tikzcd} \qquad \begin{tikzcd} X \times Y \arrow[r, "\eta_X \times \mathrm{id}"] \arrow[dr, "\eta_{X \times Y}"'] & PX \times Y \arrow[d, "\tau_{X,Y}"] \\ & P(X \times Y) \end{tikzcd}$$

$$\begin{tikzcd} PX \times (Y \times Z) \arrow[r, "\beta"] \arrow[d, "\tau_{X, Y \times Z}"'] & (PX \times Y) \times Z \arrow[r, "\tau_{X,Y} \times \mathrm{id}"] & P(X \times Y) \times Z \arrow[d, "\tau_{X \times Y, Z}"] \\ P(X \times (Y \times Z)) \arrow[rr, "P\beta"] & & P((X \times Y) \times Z) \end{tikzcd}$$

$$\begin{tikzcd} P^2 X \times Y \arrow[r, "\tau_{PX,Y}"] \arrow[d, "\mu_X \times \mathrm{id}"'] & P(PX \times Y) \arrow[r, "P\tau_{X,Y}"] & P^2(X \times Y) \arrow[d, "\mu_{X \times Y}"] \\ PX \times Y \arrow[rr, "t_{X,Y}"] & & P(X \times Y) \end{tikzcd}$$

Fig. 9. τ is the dual of tensorial strength t.